

THE \mathfrak{sl}_3 WEB ALGEBRA

M. MACKAAY, W. PAN AND D. TUBBENHAUER

ABSTRACT. In this paper we use Kuperberg's \mathfrak{sl}_3 webs and Khovanov's \mathfrak{sl}_3 foams to define a new algebra K_S , which we call the \mathfrak{sl}_3 web algebra. It is the \mathfrak{sl}_3 analogue of Khovanov's arc algebra H_n .

We prove that K_S is a graded symmetric Frobenius algebra. Furthermore, we categorify an instance of q -skew Howe duality, which allows us to prove that K_S is Morita equivalent to a certain cyclotomic KLR-algebra. This allows us to determine the Grothendieck group $K_0(K_S)$, to show that its center is isomorphic to the cohomology ring of a certain Spaltenstein variety, and to prove that K_S is a graded cellular algebra.

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1. INTRODUCTION

In this paper, we define the \mathfrak{sl}_3 analogue of Khovanov's arc algebras H_n , introduced in [26]. We call them *web algebras* and denote them by K_S , where S is a sign string (string of $+$ and $-$ signs). Instead of arc diagrams, which give a diagrammatic presentation of the representation theory of $U_q(\mathfrak{sl}_2)$, we use \mathfrak{sl}_3 webs, introduced by Kuperberg in [36]. These webs give a diagrammatic presentation of the representation theory of $U_q(\mathfrak{sl}_3)$. Instead of \mathfrak{sl}_2 cobordisms, which Bar-Natan used in [1] to give his formulation of Khovanov's link homology, we use Khovanov's [27] \mathfrak{sl}_3 foams.

We prove the following main results regarding K_S :

- (1) K_S is a graded symmetric Frobenius algebra (Theorem 3.0.24).
- (2) We give an explicit degree preserving algebra isomorphism between the cohomology ring of the Spaltenstein variety X_μ^λ and K_S , where λ and μ are two weights determined by S (Theorem 4.2.3).
- (3) Let $V_S := V_{s_1} \otimes \cdots \otimes V_{s_n}$, where V_+ is the basic $U_q(\mathfrak{sl}_3)$ representation and V_- its dual. Kuperberg [36] proved that W_S , the space of \mathfrak{sl}_3 webs whose boundary is determined by S , is isomorphic to $\text{Inv}_{U_q(\mathfrak{sl}_3)}(V_S)$, the space of invariant tensors in V_S .

Choose an arbitrary $k \in \mathbb{N}$ and let $n = 3k$. By q -skew Howe duality, which we will explain at the beginning of Section 5, we know that

$$V_{(3^k)} \cong \bigoplus_S W_S.$$

Here $V_{(3^k)}$ denotes the irreducible $U_q(\mathfrak{sl}_n)$ -module with highest weight (3^k) . The direct sum on the right-hand side is taken over all *enhanced sign sequences* of length n , which are in bijective correspondence to the semi-standard Young tableaux with k rows and 3 columns.

In Section 5 we categorify this result. Let $R_{(3^k)}$ be the cyclotomic Khovanov-Lauda-Rouquier algebra (cyclotomic KLR algebra for short) with highest weight (3^k) . Brundan and Kleshchev [6] (see also [25], [38], [58] and [59]) proved that

$$K_0(R_{(3^k)}\text{-pmod}_{\text{gr}}) \cong V_{(3^k)}^{\mathbb{Z}},$$

where the latter is the integral form of $V_{(3^k)}$.

We prove (Proposition 5.3.7) that there exists an exact degree preserving categorical $\mathcal{U}(\mathfrak{sl}_n)$ -action on

$$\bigoplus_S K_S\text{-mod}_{\text{gr}},$$

where $\mathcal{U}(\mathfrak{sl}_n)$ is Khovanov and Lauda's diagrammatic categorification of $\dot{\mathcal{U}}(\mathfrak{sl}_n)$. This categorical action can be restricted to

$$\bigoplus_S K_S\text{-pmod}_{\text{gr}}.$$

By a general result due to Rouquier [51], which we recall in Proposition 2.3.15, we get

$$(1.0.1) \quad R_{(3^k)\text{-pmod}_{\text{gr}}} \cong \bigoplus_S K_S\text{-pmod}_{\text{gr}}.$$

- (4) In particular, this proves that the split Grothendieck groups of both categories are isomorphic (Corollary 5.3.9). It follows that we have

$$K_0(K_S\text{-pmod}_{\text{gr}}) \cong W_S^{\mathbb{Z}},$$

for any S . Again, the superscript \mathbb{Z} denotes the integral form.

- (5) As proved in Corollary 5.3.9), the equivalence in (1.0.1) implies that $R_{(3^k)}$ and $\bigoplus_S K_S$ are Morita equivalent (Proposition 5.3.10), i.e. we have

$$(1.0.2) \quad R_{(3^k)\text{-mod}_{\text{gr}}} \cong \bigoplus_S K_S\text{-mod}_{\text{gr}}.$$

- (6) In Corollary 5.3.13, we show that (1.0.2) implies that K_S is a graded cellular algebra, for any S .

- (7) We show that the graded indecomposable projective K_S -modules correspond to the dual canonical basis elements in $\text{Inv}(V_S)$ (Theorem 5.3.22).

The first result is easy to prove and similar to the case for H_n . Some of the other results are much harder to prove for K_S than their analogues are for H_n (e.g. see Remark 5.3.16). In order to prove the second and the last result, we introduce a “new trick”: we use a deformation of K_S , called G_S . This deformation is induced by Gornik’s [21] deformation of Khovanov’s original \mathfrak{sl}_3 foam relations. One big difference between G_S and K_S is that the former algebra is filtered whereas the latter is graded. As a matter of fact, K_S is the associated graded algebra of G_S . The usefulness of G_S relies on the fact that G_S is semisimple as an algebra, i.e. forgetting the filtration (see Proposition 3.0.28).

Let us explain the connection to existing work in the literature. There are two diagrammatic approaches which give \mathfrak{sl}_3 link homologies: there is Khovanov’s original approach using foams [27] and there is Webster’s approach using a generalization of the cyclotomic KLR-algebras [59, 60]. In Proposition 4.4 in [60], Webster proved that both link homologies are isomorphic, but the proof is quite sophisticated and relies on Mazorchuk and Stroppel’s approach to link homology using functors and natural transformations on certain blocks of category \mathcal{O} [46]. Our results in this paper might help to give an elementary and direct proof that Khovanov and Webster’s \mathfrak{sl}_3 link homologies are isomorphic.

As we explain in more detail below, it should not be too hard to generalize our results in this paper to the case for \mathfrak{sl}_n , with $n \geq 2$, using matrix factorizations instead of foams. This could be helpful to show that Webster’s \mathfrak{sl}_n link homology is isomorphic to Khovanov and Rozansky’s link homology [34]. For $n \geq 4$, Webster has conjectured this result to hold, but he has not proved it (see his remarks below Proposition 4.4 in [60]).

Let us sketch the definition of the \mathfrak{sl}_n web algebras, for $n > 3$. For any string $S = (s_1, \dots, s_m)$, such that $1 \leq s_i \leq n - 1$, one can define a web space W_S^n . In a forthcoming paper [12], Cautis, Kamnitzer and Morrison show how to define W_S^n in terms of generators and relations. Fontaine [16] has constructed an \mathfrak{sl}_n web basis B_S^n of W_S^n which generalizes Kuperberg’s [36] basis of non-elliptic \mathfrak{sl}_3 webs (Kuperberg’s basis is recalled in Subsection 2.1). To any $w \in B_S^n$ one

can associate the colored Khovanov-Rozansky matrix factorization M_w , as defined by Wu [62] and Yonezawa [63]. For any $u, v \in B_S^n$, one can then define

$${}_u K_v^n := \text{Ext}(M_u, M_v).$$

The multiplication in

$$K_S^n := \bigoplus_{u, v \in B_S^n} {}_u K_v^n$$

is induced by the composition of homomorphisms of matrix factorizations. Proving the analogues of the results in this paper for $n > 3$ should be doable, but only after working one's way through a substantial amount of technicalities that are beyond the scope of this paper. Note that for \mathfrak{sl}_3 , the definition using matrix factorizations indeed gives an algebra isomorphic to K_S , as follows from the equivalence between matrix factorizations and foams for \mathfrak{sl}_3 proved in [43].

Another question is how $K_S\text{-pmod}_{\text{gr}}$ is related to (a subcategory of) $R_S\text{-mod}_{\text{gr}}$, where R_S is Webster's [59] generalization of the cyclotomic KLR-algebra which categorifies V_S . In the latest version of [59], Webster has added a section (Section 4.3) on the categorification of skew Howe duality within his framework of generalized cyclotomic KLR-algebras.

In [17], Fontaine, Kamnitzer and Kuperberg study spiders using an algebro-geometric approach. For \mathfrak{sl}_3 these spiders are exactly the webs in our paper. Given a sign string S , the *Satake fiber* $F(S)$, denoted $F(\vec{\lambda})$ in [17], is isomorphic to the Spaltenstein variety X_μ^λ mentioned above. Here, we point out the difference in these notations that otherwise might confuse the reader: the λ in [17] is equal to μ in our paper, which is also equivalent to S . Given a web w with boundary corresponding to S , Fontaine, Kamnitzer and Kuperberg also define a variety $Q(D(w))$, called the *web variety*. One obvious question is the following (asked to us by Kamnitzer):

Question 1.0.1. For any two basis webs $u, v \in B_S$, does there exist a degree preserving algebra isomorphism

$$H^*(Q(D(u))) \otimes_{F(S)} H^*(Q(D(v))) \cong {}_u K_v?$$

Here

$$K_S = \bigoplus_{u, v \in B_S} {}_u K_v$$

is the decomposition of K_S in Section 3 and the product on

$$\bigoplus_{u, v \in B_S} H^*(Q(D(u))) \otimes_{F(S)} H^*(Q(D(v)))$$

is given by convolution.

If the answer to this question is affirmative, then that would be the \mathfrak{sl}_3 analogue of the result, due to Stroppel and Webster [55], relating H_n to the intersection cohomology of the (n, n) -Springer fiber. Our Theorem 4.2.3 is a first step towards proving Kamnitzer's conjecture. We also note that, in [28], Khovanov showed that the center of H_n is isomorphic to the ordinary cohomology of the (n, n) -Springer fiber, before Stroppel and Webster proved the more general result.

This paper is organized as follows:

- (1) In Section 2, we recall the definitions and some fundamental properties of webs, foams and categorified quantum algebras and their categorical representations. The reader who already knows all this material well enough can just leaf through it, in order to understand our notations and conventions. Other readers might perhaps find it helpful as a brief introduction to the rapidly growing literature on categorification, although it is far from self-contained.
- (2) In Section 3, we define K_S and prove the first of our aforementioned main results.
- (3) In Section 4, we first study the relation between column strict tableaux and webs with flows. Using this relation, we prove our second main result.
- (4) In Section 5, we explain Howe duality in our context and categorify the case relevant to this paper. This leads to the other main results.
- (5) Sections 4 and 5 are largely independent of each other. However, the proof of Theorem 4.2.3 requires Proposition 5.3.10 and the proof of Proposition 5.3.19, which is a key ingredient for the proof of Theorem 5.3.22, requires Lemma 4.2.2.
- (6) In Appendix A, we collect some technical facts from the literature on filtered algebras, filtered modules and their associated graded counterparts. These are needed at various places in the paper.

2. BASIC DEFINITIONS AND BACKGROUND

2.1. Webs. In [36], Kuperberg describes the representation theory of $U_q(\mathfrak{sl}_3)$ using oriented trivalent graphs, possibly with boundary, called *webs*. Boundaries of webs consist of univalent vertices (the ends of oriented edges), which we will usually put on a horizontal line (or various horizontal lines), e.g.:

(2.1.1)

We say that a web has n free strands if the number of non-trivalent vertices is exactly n . In this way, the boundary of a web can be identified with a *sign string* $S = (s_1, \dots, s_n)$, with $s_i = \pm$, such that upward oriented boundary edges get a “+” and downward oriented boundary edges a “−” sign. Webs without boundary are called *closed webs*.

Any web can be obtained from the following elementary webs by gluing and disjoint union:

(2.1.2)

Fixing a boundary S , we can form the $\mathbb{C}(q)$ -vector space W_S , spanned by all webs with boundary S , modulo the following set of local relations (due to Kuperberg [36]):

(2.1.3)

(2.1.4)

(2.1.5)

Recall that

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} = q^{a-1} + q^{a-3} + \dots + q^{-(a-1)} \in \mathbb{N}[q, q^{-1}]$$

denotes the *quantum integer*.

By a slight abuse of jargon, we will call all elements of W_S webs. From relations (2.1.3), (2.1.4) and (2.1.5) it follows that any element in W_S is a linear combination of webs with the same boundary and without circles, digons or squares. These are called *non-elliptic webs*. As a matter of fact, the non-elliptic webs form a basis of W_S , which we call B_S . Therefore, in this paper we will simply call them *basis webs*.

Let $W_S^{\mathbb{Z}}$ be the free $\mathbb{Z}[q, q^{-1}]$ submodule of W_S generated by B_S . We call this the *integral form* of the web space.

Following Brundan and Stroppel's [9] notation for arc diagrams, we will write w^* to denote the web obtained by reflecting a given web w horizontally and reversing all orientations:

(2.1.6) 

By uv^* , we mean the planar diagram containing the disjoint union of u and v^* , where u lies vertically above v^* :

(2.1.7) 

By v^*u , we shall mean the closed web obtained by gluing v^* on top of u , when such a construction is possible (i.e. the number of free strands and orientations on the strands match):

(2.1.8) 

In the same vein, by $v_1^*u_1v_2^*u_2$ we denote the following web:

(2.1.9) 

To make the connection with the representation theory of $U_q(\mathfrak{sl}_3)$, we recall that a sign string $S = (s_1, \dots, s_n)$ corresponds to

$$V_S = V_{s_1} \otimes \dots \otimes V_{s_n},$$

where V_+ is the fundamental representation and V_- its dual. The latter is also isomorphic to $V_+ \wedge V_+$, a fact which we will need later on. Both V_+ and V_- have dimension three. Khovanov and Kuperberg [29] use a particular basis for V_+ , denoted $\{e_1^+, e_2^+, e_3^+\}$, and also one for V_- , denoted $\{e_1^-, e_2^-, e_3^-\}$. In this interpretation, webs correspond to intertwiners and

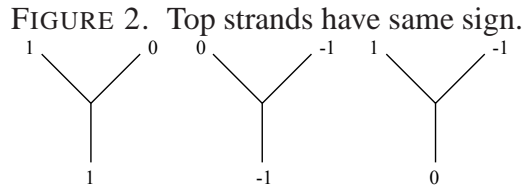
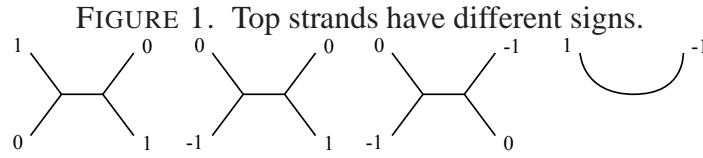
$$W_S \cong \text{Inv}(V_S).$$

Therefore, the elements of B_S give a basis of $\text{Inv}(V_S)$. However, this basis is not equal to the usual tensor basis. In Theorem 2 of [29], Kuperberg and Khovanov prove an important result concerning the change of basis matrix, which we will reproduce in Theorem 2.1.5.

Kuperberg showed in [36] (see also [29]) that basis webs are indexed by closed weight lattice paths in the dominant Weyl chamber of \mathfrak{sl}_3 . It is well-known that any path in the \mathfrak{sl}_3 -weight lattice can be presented by a pair consisting of a sign string $S = (s_1, \dots, s_n)$ and a *state string* $J = (j_1, \dots, j_n)$, with $j_i \in \{-1, 0, 1\}$ for all $1 \leq i \leq n$. Given a pair (S, J) representing a closed dominant path, a unique basis web (up to isotopy) is determined by a set of inductive rules called the *growth algorithm*. We briefly recall the algorithm as described in [29]. In fact, the algorithm can be applied to any path, but in this paper we will only use it for closed dominant paths.

Definition 2.1.1. (The Growth Algorithm) Given (S, J) , a web w_J^S is recursively generated by the following rules:

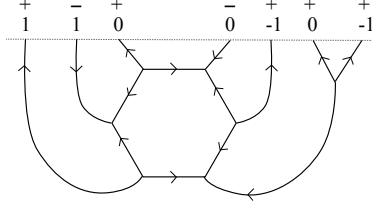
- (1) Initially, the web consists of n parallel strands whose orientations are given by the sign string. If $s_i = +$, then the i -th strand is oriented upwards; if $s_i = -$, it is oriented downwards.
- (2) The algorithm builds the web downwards. Suppose we have already applied the algorithm $k - 1$ times. For the k -th step, do the following. If the bottom boundary string contains a neighboring pair of edges matching the top of one of the following webs (called H, arc and Y respectively), then glue the corresponding H, arc or Y to the relevant bottom boundary edges:



These rules apply for any compatible orientation of the edges in the webs. Therefore, we have drawn them without any specific orientations. Below, whenever we write down an equation involving webs without orientations, we mean that the equation holds for all possible orientations. For future use, we will call the rules above the *H*, *arc* and *Y*-rule. The growth algorithm stops if no further rules can be applied.

If (S, J) represents a closed dominant path, then the growth algorithm produces a basis web.

For example, the growth algorithm converts $S = (+ - + - + +)$ and $J = (1, 1, 0, 0, -1, 0, -1)$ into the following basis web:



(2.1.10)

In addition, the growth algorithm has an inverse, called the *minimal cut path algorithm* [29], which we will not use in this paper.

Following Khovanov and Kuperberg in [29], we define a *flow* f on a web w to be an oriented subgraph that contains exactly two of the three edges incident to each trivalent vertex. The connected components of the flow are called the *flow lines*. The orientation of the flow lines need not agree with the orientation of w . Note that if w is closed, then each flow line is a closed cycle. At the boundary, the flow lines can be represented by a state string J . By convention, at the i -th boundary edge, we set $j_i = +1$ if the flow line is oriented upward, $j_i = -1$ if the flow line is oriented downward and $j_i = 0$ there is no flow line. The same convention determines a state for each edge of w .

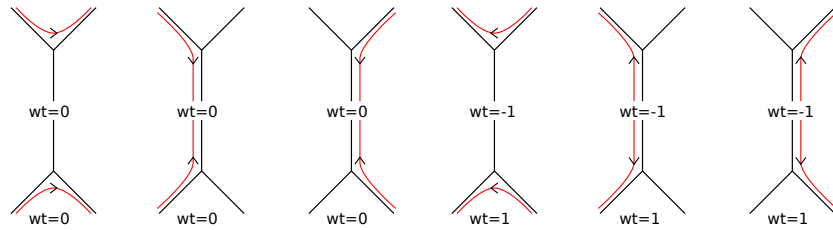
Remark 2.1.2. Every flow determines a unique 3-coloring of w , with colors $-1, 0, 1$, satisfying the property that, for any trivalent vertex of w , the colors of the three incident edges are all distinct. These colorings are called *admissible* in [21].

Conversely, any such 3-coloring determines a unique flow on w . This correspondence determines a bijection between flows and admissible 3-colorings on w .

This remark will be important in Section 5.

We will also say that any flow f that is compatible with a given state string J on the boundary of w *extends* J .

Given a web with a flow, denoted w_f , Khovanov and Kuperberg [29] attribute a *weight* to each trivalent vertex and each arc in w_f , as in Figures 2.1.11 and 2.1.12. The total weight of w_f is by definition the sum of the weights at all trivalent vertices and arcs.

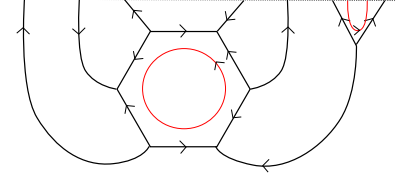


(2.1.11)



(2.1.12)

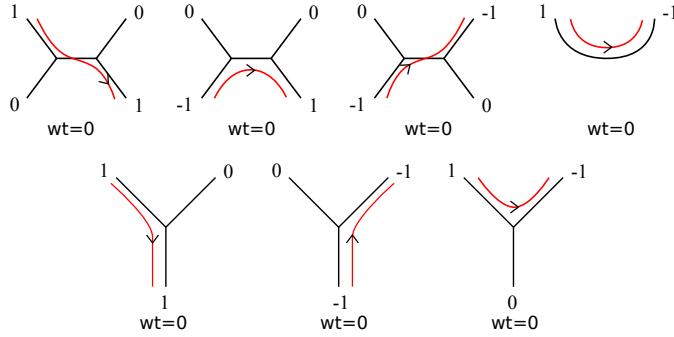
For example, the following web has weight -3 :



(2.1.13)

We can extend the table in (2.1.11) and (2.1.12) to calculate weights determined by flows on H 's, so that it becomes easier to compute the weight of w_f when w is expressed using the growth algorithm (Def. 2.1.1).

Definition 2.1.3. [29] (**Canonical Flows on Basis Webs**) Given a basis web w expressed using the growth algorithm. We define the *canonical flow* on w by the following rules:



(2.1.14)

The canonical flow does not depend on the particular instance of the growth algorithm that we have chosen to obtain w .

Observe that the definition of the canonical flows imply the following lemma.

Lemma 2.1.4. *A basis web with the canonical flow has weight zero.*

Given (S, J) , let

$$e_J^S := e_{j_1}^{s_1} \otimes \cdots \otimes e_{j_n}^{s_n}.$$

Khovanov and Kuperberg prove the following theorem (Theorem 2 in [29]), which will be important for us in Section 5:

Theorem 2.1.5 (Khovanov-Kuperberg). *Given (S, J) , we have*

$$w_J^S = e_J^S + \sum_{J' < J} c(S, J, J') e_{J'}^S,$$

for some coefficients $c(S, J, J') \in \mathbb{N}[q, q^{-1}]$, where the state strings J and J' are ordered lexicographically.

Remark 2.1.6. Khovanov and Kuperberg [29] show that B_S is not equal to the dual canonical basis of W_S . This follows from the fact that $c(S, J, J') \notin q^{-1}\mathbb{N}[q^{-1}]$ in general. In their Section 8, they give explicit counter-examples of elements $w \in B_S$ which admit non-canonical weight zero flows.

2.2. Foams. In this subsection we review the category \mathbf{Foam}_3 of \mathfrak{sl}_3 -foams introduced by Khovanov in [27]. As a matter of fact, we will also need a deformation of Khovanov's original category, due to Gornik [21] in the context of matrix factorizations, and studied in [42] in the context of foams. Therefore, we introduce a parameter $c \in \mathbb{C}$ in \mathbf{Foam}_3^c , just as in [42], such that we get Khovanov's original category for $c = 0$ and, for any $c \neq 0$, the category \mathbf{Foam}_3^c is isomorphic to Gornik's deformation (his original deformation was for $c = 1$). A big difference between these two specializations is that \mathbf{Foam}_3^c is graded for $c = 0$ and filtered for any $c \neq 0$. In fact, for any $c \neq 0$, the associated graded category of \mathbf{Foam}_3^c is isomorphic to \mathbf{Foam}_3^0 .

We recall the following definitions as they appear in [42]. We note that the diagrams accompanying these definitions are taken, also, from [42].

A *pre-foam* is a cobordism with singular arcs between two webs. A singular arc in a pre-foam U is the set of points of U which have a neighborhood homeomorphic to the letter Y times an interval. Note that singular arcs are disjoint. Interpreted as morphisms, we read pre-foams from bottom to top by convention. Thus, pre-foam composition consists of placing one pre-foam on top of the other. The orientation of the singular arcs is is, by convention, as in the diagrams below (called the *zip* and the *unzip* respectively):



We allow pre-foams to have dots that can move freely about the facet on which they belong, but we do not allow dot to cross singular arcs.

By a *foam*, we mean a formal \mathbb{C} -linear combination of isotopy classes of pre-foams modulo the ideal generated by the set of relations $\ell = (3D, NC, S, \Theta)$ and the *closure relation*, as described below.

$$\begin{aligned}
(3D) \quad & \text{A parallelogram with three dots} = c \text{ A parallelogram} \\
(NC) \quad & \text{A cylinder} = - \text{A cup with two dots} - \text{A cup with one dot} - \text{A cup with no dots} \\
(S) \quad & \text{A sphere with no dots} = \text{A sphere with one dot} = 0, \quad \text{A sphere with two dots} = -1 \\
(\Theta) \quad & \text{A sphere with three dots labeled } \alpha, \beta, \delta = \begin{cases} 1, & (\alpha, \beta, \delta) = (1, 2, 0) \text{ or a cyclic permutation,} \\ -1, & (\alpha, \beta, \delta) = (2, 1, 0) \text{ or a cyclic permutation,} \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

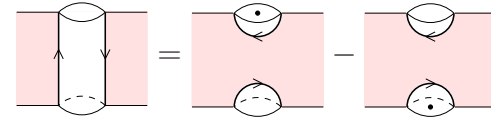
The *closure relation*: any \mathbb{C} -linear combination of foams, with the same boundary, is equal to zero if and only if any way of capping off these foams with a common foam yields a \mathbb{C} -linear combination of closed foams whose evaluation is zero.

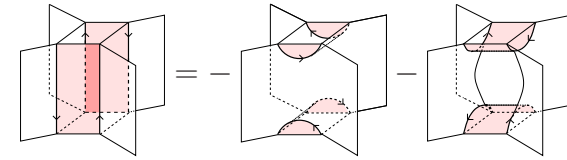
The relations in ℓ imply the following identities (for detailed proofs see [27]).

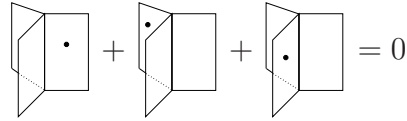
(Bamboo) 

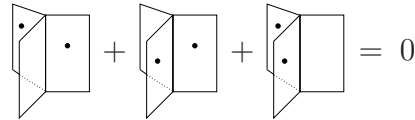
(RD) 

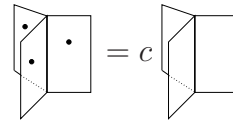
(Bubble) 

(DR) 

(SqR) 



(Dot Migration) 



Definition 2.2.1. For any $c \in \mathbb{C}$, let \mathbf{Foam}_3^c be the category whose objects are webs Γ lying inside a horizontal strip in \mathbb{R}^2 , which is bounded by the lines $y = 0, 1$ containing the boundary points of Γ . The morphisms of \mathbf{Foam}_3^c are \mathbb{C} -linear combinations of foams lying inside the horizontal strip bounded by $y = 0, 1$ times the unit interval. We require that the vertical boundary of each foam is a set (possibly empty) of vertical lines.

The q -grading of a foam U is defined as

$$q(U) := \chi(\partial U) - 2\chi(U) + 2d + b,$$

where χ denotes the Euler characteristic, d is the number of dots on U and b is the number of vertical boundary components. This makes \mathbf{Foam}_3^0 into a graded category. For any $c \neq 0$, this makes \mathbf{Foam}_3^c into a filtered category, whose associated graded category is isomorphic to \mathbf{Foam}_3^0 .

Definition 2.2.2. [27] (**Foam Homology**) Given a web w the *foam homology* of w is the complex vector space, $\mathcal{F}^c(w)$, spanned by all foams

$$U : \emptyset \rightarrow w$$

in Foam_3^c .

The complex vector space $\mathcal{F}^c(w)$ is filtered/graded by the q -grading on foams and has rank $\langle w \rangle$, where $\langle w \rangle$ is the *Kuperberg bracket* computed recursively by the rules:

- (1) $\langle w \amalg \bigcirc \rangle = [3] \langle w \rangle$,
- (2) $\langle \begin{array}{c} \longrightarrow \\ \circlearrowleft \end{array} \rangle = [2] \langle \longrightarrow \rangle$,
- (3) $\langle \begin{array}{c} \longrightarrow \\ \square \\ \longrightarrow \end{array} \rangle = \langle \longrightarrow \rangle \langle \longrightarrow \rangle + \langle \begin{array}{c} \longrightarrow \\ \curvearrowright \end{array} \rangle$.

The relations above correspond to the decomposition of $\mathcal{F}^c(w)$ into direct summands. The idempotents corresponding to these direct summands are the terms on the r.h.s. of the relations (NC), (DR) and (SqR), respectively. For any $c \neq 0$, the complex vector space $\mathcal{F}^c(w)$ is filtered and its associated graded vector space is $\mathcal{F}^0(w)$. See [26, 42] for details.

Remark 2.2.3. Given $u, v \in B_S$, the observations above and Theorem 2.1.5 show that there exists a homogeneous basis of $\mathcal{F}^0(u^*v)$ parametrized by the flows on u^*v . We have, in fact, constructed such a basis, but it is not unique. There is also no “preferred choice”, unless one requires the basis to have other nice properties, e.g. in the \mathfrak{sl}_2 case, Brundan and Stroppel prove that there is a cellular basis of H_n . The construction of a “good” basis of the \mathfrak{sl}_3 web algebra K_S (and similarly for Gornik’s deformation G_S) is still work in progress and will, hopefully, be the contents of a subsequent paper. Although we do not need such a basis in this paper, it is important that the reader keep this remark in mind while reading Section 5.

2.3. Quantum 2-algebras.

2.3.1. The quantum general and special linear algebras. First we recall the quantum general and special linear algebras. Most parts in this section are copied from section two and three in [41]. Note that, in contrast to [41], we work over $\mathbb{C}(q)$ instead of $\mathbb{Q}(q)$.

The \mathfrak{gl}_n -weight lattice is isomorphic to \mathbb{Z}^n . Let $\epsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$, with 1 being on the i -th coordinate, and $\alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \dots, 1, -1, \dots, 0) \in \mathbb{Z}^n$, for $i = 1, \dots, n-1$. Recall that the Euclidean inner product on \mathbb{Z}^n is defined by $(\epsilon_i, \epsilon_j) = \delta_{i,j}$.

Definition 2.3.1. For $n \in \mathbb{N}_{>1}$ the *quantum general linear algebra* $U_q(\mathfrak{gl}_n)$ is the associative unital $\mathbb{C}(q)$ -algebra generated by K_i and K_i^{-1} , for $1, \dots, n$, and $E_{\pm i}$, for $i = 1, \dots, n-1$, subject to the relations

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ E_i E_{-j} - E_{-j} E_i &= \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}, \\ K_i E_{\pm j} &= q^{\pm(\epsilon_i, \alpha_j)} E_{\pm j} K_i, \\ E_{\pm i}^2 E_{\pm j} - (q + q^{-1}) E_{\pm i} E_{\pm j} E_{\pm i} + E_{\pm j} E_{\pm i}^2 &= 0, & \text{if } |i - j| = 1, \\ E_{\pm i} E_{\pm j} - E_{\pm j} E_{\pm i} &= 0, & \text{else.} \end{aligned}$$

Definition 2.3.2. For $n \in \mathbb{N}_{>1}$ the *quantum special linear algebra* $U_q(\mathfrak{sl}_n) \subseteq U_q(\mathfrak{gl}_n)$ is the unital $\mathbb{C}(q)$ -subalgebra generated by $K_i K_{i+1}^{-1}$ and $E_{\pm i}$, for $i = 1, \dots, n-1$.

Recall that the $U_q(\mathfrak{sl}_n)$ -weight lattice is isomorphic to \mathbb{Z}^{n-1} . Suppose that V is a $U_q(\mathfrak{gl}_n)$ -weight representation with weights $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_{\lambda} V_{\lambda},$$

and K_i acts as multiplication by q^{λ_i} on V_{λ} . Then V is also a $U_q(\mathfrak{sl}_n)$ -weight representation with weights $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\bar{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j = 1, \dots, n-1$.

Conversely, given a $U_q(\mathfrak{sl}_n)$ -weight representation with weights $\mu = (\mu_1, \dots, \mu_{n-1})$, there is not a unique choice of $U_q(\mathfrak{gl}_n)$ -action on V . We can fix this by choosing the action of $K_1 \cdots K_n$. In terms of weights, this corresponds to the observation that, for any $d \in \mathbb{Z}$, the equations

$$(2.3.1) \quad \lambda_i - \lambda_{i+1} = \mu_i,$$

$$(2.3.2) \quad \sum_{i=1}^n \lambda_i = d$$

determine $\lambda = (\lambda_1, \dots, \lambda_n)$ uniquely, if there exists a solution to (2.3.1) and (2.3.2) at all. To fix notation, we define the map $\phi_{n,d}: \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^n \cup \{*\}$ by

$$\phi_{n,d}(\mu) = \lambda,$$

if (2.3.1) and (2.3.2) have a solution, and we put $\phi_{n,d}(\mu) = *$ otherwise.

Note that $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$ are both Hopf algebras, which implies that the tensor product of two of their representations is a representation again.

Both $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$ have plenty of non-weight representations, but we won't discuss them in the paper. Therefore we can restrict our attention to the Beilinson-Lusztig-MacPherson [2] idempotent version of these quantum groups, denoted $\dot{U}(\mathfrak{gl}_n)$ and $\dot{U}(\mathfrak{sl}_n)$ respectively. To understand their definition, recall that K_i acts as q^{λ_i} on the λ -weight space of any weight representation. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_{λ} to $U_q(\mathfrak{gl}_n)$ and add the relations

$$\begin{aligned} 1_{\lambda} 1_{\mu} &= \delta_{\lambda, \nu} 1_{\lambda}, \\ E_{\pm i} 1_{\lambda} &= 1_{\lambda \pm \alpha_i} E_{\pm i}, \\ K_i 1_{\lambda} &= q^{\lambda_i} 1_{\lambda}. \end{aligned}$$

Definition 2.3.3. The idempotent quantum general linear algebra is defined by

$$\dot{U}(\mathfrak{gl}_n) = \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} 1_{\lambda} U_q(\mathfrak{gl}_n) 1_{\mu}.$$

Let $I = \{1, 2, \dots, n-1\}$. In the sequel we use *signed sequences* $\underline{i} = (\alpha_1 i_1, \dots, \alpha_m i_m)$, for any $m \in \mathbb{N}$, $\alpha_j \in \{\pm 1\}$ and $i_j \in I$. The set of signed sequences we denote SiSeq .

For such an $\underline{i} = (\alpha_1 i_1, \dots, \alpha_{n-1} i_{n-1})$ we define

$$E_{\underline{i}} := E_{\alpha_1 i_1} \cdots E_{\alpha_{n-1} i_{n-1}}$$

and we define $\underline{i}_{\Lambda} \in \mathbb{Z}^n$ to be the n -tuple such that

$$E_{\underline{i}} 1_{\mu} = 1_{\mu + \underline{i}_{\Lambda}} E_{\underline{i}}.$$

Similarly, for $U_q(\mathfrak{sl}_n)$, adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_\mu 1_\nu &= \delta_{\mu,\nu} 1_\lambda, \\ E_{\pm i} 1_\mu &= 1_{\mu \pm \bar{\alpha}_i} E_{\pm i}, \quad \text{with } \bar{\alpha}_i = \alpha_i - \alpha_{i+1}, \\ K_i K_{i+1}^{-1} 1_\mu &= q^{\mu_i} 1_\mu. \end{aligned}$$

Definition 2.3.4. The idempotent quantum special linear algebra is defined by

$$\dot{U}(\mathfrak{sl}_n) = \bigoplus_{\mu, \nu \in \mathbb{Z}^{n-1}} 1_\mu U_q(\mathfrak{sl}_n) 1_\nu.$$

Note that $\dot{U}(\mathfrak{gl}_n)$ and $\dot{U}(\mathfrak{sl}_n)$ are both non-unital algebras, because their units would have to be equal to the infinite sum of all their idempotents. Furthermore, the only $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$ -representations which factor through $\dot{U}(\mathfrak{gl}_n)$ and $\dot{U}(\mathfrak{sl}_n)$ respectively are the weight representations. Finally, note that there is no embedding of $\dot{U}(\mathfrak{sl}_n)$ into $\dot{U}(\mathfrak{gl}_n)$, because there is no embedding of the \mathfrak{sl}_n -weights into the \mathfrak{gl}_n -weights.

Finally, recall the *integral forms* of these quantum algebras. For each $i = 1, \dots, n-1$ and each $a \in \mathbb{N}$, define the *divided power*

$$E_{\pm i}^{(a)} := \frac{E_{\pm i}^a}{[a]}.$$

Definition 2.3.5. Let $\dot{U}^{\mathbb{Z}}(\mathfrak{gl}_n) \subset \dot{U}(\mathfrak{gl}_n)$ and $\dot{U}^{\mathbb{Z}}(\mathfrak{sl}_n) \subset \dot{U}(\mathfrak{sl}_n)$ be the $\mathbb{Z}[q, q^{-1}]$ subalgebras generated by the divided powers $E_{\pm i}^{(a)} 1_\lambda$.

2.3.2. The q -Schur algebra. Let $d \in \mathbb{N}$ and let V be the natural n -dimensional representation of $U_q(\mathfrak{gl}_n)$. Define

$$\Lambda(n, d) = \{\lambda \in \mathbb{N}^n \mid \sum_{i=1}^n \lambda_i = d\} \quad \text{and}$$

$$\Lambda^+(n, d) = \{\lambda \in \Lambda(n, d) \mid d \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}.$$

Recall that the weights in $V^{\otimes d}$ are precisely the elements of $\Lambda(n, d)$, and that the highest weights are the elements of $\Lambda^+(n, d)$. The highest weights correspond exactly to the irreducibles V_λ that show up in the decomposition of $V^{\otimes d}$.

We can define the q -Schur algebra as follows:

Definition 2.3.6. The q -Schur algebra $S_q(n, d)$ is the image of the representation $\psi_{n,d}$ defined by

$$\psi_{n,d}: U_q(\mathfrak{gl}_n) \rightarrow \text{End}_{\mathbb{C}}(V^{\otimes d}).$$

For $\lambda \in \Lambda^+(n, d)$, the $U_q(\mathfrak{gl}_n)$ -action on V_λ factors through the projection $\psi_{n,d}: U_q(\mathfrak{gl}_n) \rightarrow S_q(n, d)$. This way we obtain all irreducible representations of $S_q(n, d)$. Note that this also implies that all representations of $S_q(n, d)$ have a weight decomposition. As a matter of fact, it is well-known that

$$S_q(n, d) \cong \prod_{\lambda \in \Lambda^+(n, d)} \text{End}_{\mathbb{C}}(V_\lambda).$$

Therefore $S_q(n, d)$ is a finite-dimensional semi-simple unital algebra and its dimension is equal to

$$\sum_{\lambda \in \Lambda^+(n, d)} \dim(V_\lambda)^2 = \binom{n^2 + d - 1}{d}.$$

Since $V^{\otimes d}$ is a weight representation, $\psi_{n,d}$ gives rise to a homomorphism $\dot{\mathbf{U}}(\mathfrak{gl}_n) \rightarrow S_q(n, d)$, for which we use the same notation. This map is still surjective and Doty and Giaquinto, in Theorem 2.4 of [15], showed that the kernel of $\psi_{n,d}$ is equal to the ideal generated by all idempotents 1_λ such that $\lambda \notin \Lambda(n, d)$. Clearly the kernel of $\psi_{n,d}$ is isomorphic to $S_q(n, d)$. By the above observations, we see that $S_q(n, d)$ has a Serre presentation. As a matter of fact, by Corollary 4.3.2 in [14], this presentation is simpler than that of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$: one does not need to impose the last two Serre relations, involving cubical terms, because they are implied by the other relations and the finite dimensionality.

Lemma 2.3.7. *$S_q(n, d)$ is isomorphic to the associative unital $\mathbb{C}(q)$ -algebra generated by 1_λ , for $\lambda \in \Lambda(n, d)$, and $E_{\pm i}$, for $i = 1, \dots, n-1$, subject to the relations*

$$(2.3.3) \quad 1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda,$$

$$(2.3.4) \quad \sum_{\lambda \in \Lambda(n, d)} 1_\lambda = 1,$$

$$(2.3.5) \quad E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i}, \quad \text{with } \alpha_i = \epsilon_i - \epsilon_{i+1} = (0, \dots, 1, -1, \dots, 0),$$

$$(2.3.6) \quad E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} [\bar{\lambda}_i] 1_\lambda.$$

We use the convention that $1_\mu X 1_\lambda = 0$, if μ or λ is not contained in $\Lambda(n, d)$. Again $[a]$ denotes the q -integer from before.

Although there is no embedding of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ into $\dot{\mathbf{U}}(\mathfrak{gl}_n)$, the projection

$$\psi_{n,d}: \mathbf{U}_q(\mathfrak{gl}_n) \rightarrow S_q(n, d)$$

can be restricted to $\mathbf{U}_q(\mathfrak{sl}_n)$ and is still surjective. This gives rise to the surjection

$$\psi_{n,d}: \dot{\mathbf{U}}(\mathfrak{sl}_n) \rightarrow S_q(n, d),$$

defined by

$$(2.3.7) \quad \psi_{n,d}(E_{\pm i} 1_\lambda) = E_{\pm i} 1_{\phi_{n,d}(\lambda)},$$

where $\phi_{n,d}$ was defined below equations (2.3.1) and (2.3.2). By convention we put $1_* = 0$.

Just for completeness, let us also recall the integral form of the q -Schur algebra.

Definition 2.3.8. Define $S_q^{\mathbb{Z}}(n, d) \subset S_q(n, d)$ to be the $\mathbb{Z}[q, q^{-1}]$ subalgebra generated by the divided powers $E_{\pm i}^{(a)} 1_\lambda$.

2.3.3. The general and special quantum 2-algebras. We note that a lot of this section is copied from [41]. The reader can find even more details there.

Let $\mathcal{U}(\mathfrak{sl}_n)$ be Khovanov and Lauda's [31] diagrammatic categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$. In [41] it was shown that there is a quotient 2-category of $\mathcal{U}(\mathfrak{sl}_n)$, denoted $\mathcal{S}(n, n)$, which categorifies $S_q(n, n)$.

We recall the definition of these categorified quantum algebras and some notions from above. As before, let $I = \{1, 2, \dots, n-1\}$. Again, we use *signed sequences* $\underline{i} = (\alpha_1 i_1, \dots, \alpha_m i_m)$,

for any $m \in \mathbb{N}$, $\alpha_j \in \{\pm 1\}$ and $i_j \in I$, and the set of signed sequences is denoted SiSeq . For $\underline{i} = (\alpha_1 i_1, \dots, \alpha_m i_m) \in \text{SiSeq}$ we define $\underline{i}_\Lambda := \alpha_1 (i_1)_\Lambda + \dots + \alpha_m (i_m)_\Lambda$, where

$$(i_j)_\Lambda = (0, 0, \dots, 1, -1, 0, \dots, 0),$$

such that the vector starts with $i_j - 1$ and ends with $k - 1 - i_j$ zeros. We also define the symmetric \mathbb{Z} -valued bilinear form on $\mathbb{C}[I]$ by $i \cdot i = 2$, $i \cdot (i + 1) = -1$ and $i \cdot j = 0$, for $|i - j| > 1$. Recall that $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$.

We first recall the definition, given in [41], of the 2-category which conjecturally categorifies $\dot{\mathcal{U}}(\mathfrak{gl}_n)$. It is a straightforward adaptation of Khovanov and Lauda's $\mathcal{U}(\mathfrak{sl}_n)$.

Definition 2.3.9. $\mathcal{U}(\mathfrak{gl}_n)$ is an additive \mathbb{C} -linear 2-category. The 2-category $\mathcal{U}(\mathfrak{gl}_n)$ consists of

- Objects: $\lambda \in \mathbb{Z}^n$.

The hom-category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ between two objects λ, λ' is an additive \mathbb{C} -linear category consisting of:

- Objects¹ of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$: a 1-morphism in $\mathcal{U}(\mathfrak{gl}_n)$ from λ to λ' is a formal finite direct sum of 1-morphisms

$$\mathcal{E}_{\underline{i}} \mathbf{1}_\lambda \{t\} = \mathbf{1}_{\lambda'} \mathcal{E}_{\underline{i}} \mathbf{1}_\lambda \{t\} := \mathcal{E}_{\alpha_1 i_1} \cdots \mathcal{E}_{\alpha_m i_m} \mathbf{1}_\lambda \{t\}$$

for any $t \in \mathbb{Z}$ and signed sequence $\underline{i} \in \text{SiSeq}$ such that $\lambda' = \lambda + \underline{i}_\Lambda$ and $\lambda, \lambda' \in \mathbb{Z}^n$.

- Morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$: for 1-morphisms $\mathcal{E}_{\underline{i}} \mathbf{1}_\lambda \{t\}$ and $\mathcal{E}_{\underline{j}} \mathbf{1}_\lambda \{t'\}$ in $\mathcal{U}(\mathfrak{gl}_n)$, the hom sets $\mathcal{U}(\mathfrak{gl}_n)(\mathcal{E}_{\underline{i}} \mathbf{1}_\lambda \{t\}, \mathcal{E}_{\underline{j}} \mathbf{1}_\lambda \{t'\})$ of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are graded \mathbb{C} -vector spaces given by linear combinations of degree $t - t'$ diagrams, modulo certain relations, built from composites of:
 - (i) Degree zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}(\mathfrak{gl}_n)$; the identity 2-morphisms $1_{\mathcal{E}_{+i} \mathbf{1}_\lambda \{t\}}$ and $1_{\mathcal{E}_{-i} \mathbf{1}_\lambda \{t\}}$, for $i \in I$, are represented graphically by

$$\begin{array}{ccc} 1_{\mathcal{E}_{+i} \mathbf{1}_\lambda \{t\}} & & 1_{\mathcal{E}_{-i} \mathbf{1}_\lambda \{t\}} \\ \begin{array}{c} i \\ | \\ \lambda + i_\Lambda \bigwedge \lambda \\ | \\ i \end{array} & & \begin{array}{c} i \\ | \\ \lambda - i_\Lambda \bigvee \lambda \\ | \\ i \end{array} \\ \text{deg } 0 & & \text{deg } 0 \end{array}$$

for any $\lambda + i_\Lambda \in \mathbb{Z}^n$ and any $\lambda - i_\Lambda \in \mathbb{Z}^n$, respectively.



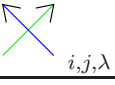
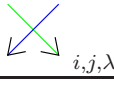


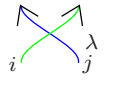

More generally, for a signed sequence $\underline{i} = (\alpha_1 i_1, \alpha_2 i_2, \dots, \alpha_m i_m)$, the identity $1_{\mathcal{E}_{\underline{i}} \mathbf{1}_\lambda \{t\}}$ 2-morphism is represented as









$$\begin{array}{c} i_1 \quad i_2 \quad \dots \quad i_m \\ | \quad | \quad \dots \quad | \\ \lambda + \underline{i}_\Lambda \quad \dots \quad \lambda \\ | \quad | \quad \dots \quad | \\ i_1 \quad i_2 \quad \dots \quad i_m \end{array}$$

where the strand labeled i_k is oriented up if $\alpha_k = +$ and oriented down if $\alpha_k = -$. We will often place labels with no sign on the side of a strand and omit the labels at the top and bottom. The signs can be recovered from the orientations on the strands.

¹We refer to objects of the category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ as 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$. Likewise, the morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are called 2-morphisms in $\mathcal{U}(\mathfrak{gl}_n)$.

(ii) Recall that $- \cdot -$ is the bilinear form from above. For each $\lambda \in \mathbb{Z}^n$ the 2-morphisms

Notation:				
2-morphism:				
Degree:	$i \cdot i$	$i \cdot i$	$-i \cdot j$	$-i \cdot j$

Notation:				
2-morphism:				
Degree:	$1 - \bar{\lambda}_i$	$1 + \bar{\lambda}_i$	$1 + \bar{\lambda}_i$	$1 - \bar{\lambda}_i$

• Biadjointness and cyclicity:

(i) $1_{\lambda+i_\Lambda} \mathcal{E}_{+i} 1_\lambda$ and $1_\lambda \mathcal{E}_{-i} 1_{\lambda+i_\Lambda}$ are biadjoint, up to grading shifts:

$$(2.3.8) \quad \begin{array}{c} \lambda+i_\Lambda \\ \text{blue arc with arrow up} \\ \lambda \end{array} = \begin{array}{c} \lambda+i_\Lambda \\ \text{blue dot on upward arrow} \\ i \end{array} \quad \begin{array}{c} \lambda \\ \text{blue arc with arrow down} \\ \lambda+i_\Lambda \end{array} = \begin{array}{c} \lambda \\ \text{blue dot on downward arrow} \\ i \end{array}$$

$$(2.3.9) \quad \begin{array}{c} \lambda \\ \text{blue arc with arrow up} \\ \lambda+i_\Lambda \end{array} = \begin{array}{c} \lambda+i_\Lambda \\ \text{blue dot on upward arrow} \\ i \end{array} \quad \begin{array}{c} \lambda+i_\Lambda \\ \text{blue arc with arrow down} \\ \lambda \end{array} = \begin{array}{c} \lambda \\ \text{blue dot on downward arrow} \\ i \end{array}$$

(ii)

$$(2.3.10) \quad \begin{array}{c} \lambda+i_\Lambda \\ \text{blue arc with arrow up} \\ \lambda \end{array} = \begin{array}{c} \lambda \\ \text{blue dot on upward arrow} \\ i \end{array} = \begin{array}{c} \lambda+i_\Lambda \\ \text{blue arc with arrow down} \\ \lambda \end{array}$$

(iii) All 2-morphisms are cyclic with respect to the above biadjoint structure. This is ensured by the relations (2.3.10), and, for arbitrary i, j , the relations

$$(2.3.11) \quad \begin{array}{c} \text{Diagram 1: A blue arc with arrow up, labeled i at the bottom and j at the top, with a green arc with arrow down, labeled j at the bottom and i at the top, crossing it.} \\ \text{Diagram 2: A crossing of two arrows, one green and one blue, labeled i and j at the bottom and lambda at the top.} \\ \text{Diagram 3: A blue arc with arrow down, labeled i at the bottom and j at the top, with a green arc with arrow up, labeled j at the bottom and i at the top, crossing it.} \end{array}$$

Note that we can take either the first or the last diagram above as the definition of the up-side-down crossing. The cyclic condition on 2-morphisms, expressed by (2.3.10) and (2.3.11), ensures that diagrams related by isotopy represent the same 2-morphism in $\mathcal{U}(\mathfrak{gl}_n)$.

It will be convenient to introduce degree zero 2-morphisms:

$$(2.3.12) \quad \begin{array}{c} i \\ \swarrow \searrow \\ j \end{array} \lambda = \begin{array}{c} i \quad j \\ | \quad | \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ j \quad i \end{array} \lambda = \begin{array}{c} i \quad j \\ | \quad | \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ j \quad i \end{array} \lambda$$

$$(2.3.13) \quad \begin{array}{c} i \\ \swarrow \searrow \\ j \end{array} \lambda = \begin{array}{c} j \quad i \\ | \quad | \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad j \end{array} \lambda = \begin{array}{c} j \quad i \\ | \quad | \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad j \end{array} \lambda,$$

where the second equality in (2.3.12) and (2.3.13) follow from (2.3.11).

(iv) All dotted bubbles of negative degree are zero. That is,

$$(2.3.14) \quad \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ m \end{array} \lambda = 0, \quad \text{if } m < \bar{\lambda}_i - 1, \quad \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ m \end{array} \lambda = 0, \quad \text{if } m < -\bar{\lambda}_i - 1$$

for all $m \in \mathbb{Z}_+$, where a dot carrying a label m denotes the m -fold iterated vertical composite of $\begin{array}{c} \uparrow \\ \bullet \\ i, \lambda \end{array}$ or $\begin{array}{c} \downarrow \\ \bullet \\ i, \lambda \end{array}$ depending on the orientation. A dotted bubble of degree zero equals ± 1 :

$$(2.3.15) \quad \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ m \end{array} \lambda = (-1)^{\lambda_{i+1}}, \quad \text{for } \bar{\lambda}_i \geq 1, \quad \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ m \end{array} \lambda = (-1)^{\lambda_{i+1}-1}, \quad \text{for } \bar{\lambda}_i \leq -1.$$

(v) For the following relations we employ the convention that all summations are increasing, so that a summation of the form $\sum_{f=0}^m$ is zero if $m < 0$.

$$(2.3.16) \quad \begin{array}{c} i \\ \swarrow \searrow \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad i \end{array} \lambda = - \sum_{f=0}^{-\bar{\lambda}_i} \begin{array}{c} \bullet \\ -\bar{\lambda}_i - f \\ | \\ i \end{array} \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ \bar{\lambda}_i - 1 + f \end{array} \lambda \quad \text{and} \quad \begin{array}{c} i \\ \swarrow \searrow \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad i \end{array} \lambda = \sum_{g=0}^{\bar{\lambda}_i} \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ -\bar{\lambda}_i - 1 + g \end{array} \lambda \begin{array}{c} \bullet \\ \bar{\lambda}_i - g \\ | \\ i \end{array}$$

$$(2.3.17) \quad \begin{array}{c} \lambda \\ \swarrow \searrow \\ i \quad i \end{array} \lambda = \begin{array}{c} i \\ \swarrow \searrow \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad i \end{array} \lambda - \sum_{f=0}^{\bar{\lambda}_i-1} \sum_{g=0}^f \begin{array}{c} \bullet \\ \bar{\lambda}_i - 1 - f \\ | \\ i \end{array} \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ -\bar{\lambda}_i - 1 + g \end{array} \lambda \quad \text{and} \quad \begin{array}{c} \lambda \\ \swarrow \searrow \\ i \quad i \end{array} \lambda = \begin{array}{c} i \\ \swarrow \searrow \\ \text{---} \text{---} \text{---} \text{---} \\ | \quad | \\ i \quad i \end{array} \lambda - \sum_{f=0}^{-\bar{\lambda}_i-1} \sum_{g=0}^f \begin{array}{c} \bullet \\ -\bar{\lambda}_i - 1 - f \\ | \\ i \end{array} \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ \bar{\lambda}_i - 1 + g \end{array} \lambda$$

for all $\lambda \in \mathbb{Z}^n$ (see (2.3.12) and (2.3.13) for the definition of sideways crossings). Notice that for some values of λ the dotted bubbles appearing above have negative labels. A composite of $\begin{array}{c} \uparrow \\ \bullet \\ i, \lambda \end{array}$ or $\begin{array}{c} \downarrow \\ \bullet \\ i, \lambda \end{array}$ with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called *fake bubbles*, are formal symbols inductively defined by the equation

$$(2.3.18) \quad \left(\begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ -\bar{\lambda}_i - 1 \end{array} \lambda t^0 + \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ -\bar{\lambda}_i - 1 + 1 \end{array} \lambda t^1 + \cdots + \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ -\bar{\lambda}_i - 1 + r \end{array} \lambda t^r + \cdots \right) \left(\begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ \bar{\lambda}_i - 1 \end{array} \lambda t^0 + \cdots + \begin{array}{c} i \\ \swarrow \searrow \\ \bullet \\ \bar{\lambda}_i - 1 + r \end{array} \lambda t^r + \cdots \right) = -1$$

and the additional condition

$$\begin{array}{c} i \\ \circlearrowleft \\ -1 \end{array} \lambda = (-1)^{\lambda_{i+1}} \quad \text{and} \quad \begin{array}{c} i \\ \circlearrowright \\ -1 \end{array} \lambda = (-1)^{\lambda_{i+1}-1}, \quad \text{if } \bar{\lambda}_i = 0.$$

Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles.

(vi) NilHecke relations:

$$(2.3.19) \quad \begin{array}{c} i \\ \circlearrowleft \\ i \end{array} \lambda = 0, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda$$

$$(2.3.20) \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \lambda - \begin{array}{c} \searrow \\ \bullet \\ \nearrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \lambda - \begin{array}{c} \searrow \\ \bullet \\ \nearrow \end{array} \lambda.$$

• For $i \neq j$:

$$(2.3.21) \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \searrow \end{array} \lambda \quad \text{and} \quad \begin{array}{c} \searrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} \lambda = \begin{array}{c} \searrow \\ \nearrow \end{array} \lambda$$

• (i) For $i \neq j$:

$$(2.3.22) \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda = \begin{cases} \begin{array}{c} \nearrow \\ \searrow \end{array} \lambda, & \text{if } i \cdot j = 0, \\ (i - j) \left(\begin{array}{c} \bullet \\ \nearrow \\ \searrow \end{array} \lambda - \begin{array}{c} \bullet \\ \searrow \\ \nearrow \end{array} \lambda \right), & \text{if } i \cdot j = -1. \end{cases}$$

Notice that $(i - j)$ is just a sign, which takes into account the standard orientation of the Dynkin diagram.

$$(2.3.23) \quad \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \lambda = \begin{array}{c} \searrow \\ \bullet \\ \nearrow \end{array} \lambda \quad \text{and} \quad \begin{array}{c} \searrow \\ \bullet \\ \nearrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} \lambda.$$

(ii) Unless $i = k$ and $i \cdot j = -1$:

$$(2.3.24) \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda = \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \lambda$$

(iii) For $i \cdot j = -1$:

$$(2.3.25) \quad \begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} = (i - j) \begin{array}{c} \text{Diagram 3} \end{array} \begin{array}{c} \text{Diagram 4} \end{array} \begin{array}{c} \text{Diagram 5} \end{array} \lambda.$$

- The additive, linear composition functor $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda') \times \mathcal{U}(\mathfrak{gl}_n)(\lambda', \lambda'') \rightarrow \mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda'')$ is given on 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by

$$(2.3.26) \quad \mathcal{E}_{\underline{j}} \mathbf{1}_{\lambda'} \{t'\} \times \mathcal{E}_{\underline{i}} \mathbf{1}_{\lambda} \{t\} \mapsto \mathcal{E}_{\underline{j}\underline{i}} \mathbf{1}_{\lambda} \{t + t'\}$$

for $\underline{i}_{\Lambda} = \lambda - \lambda'$, and on 2-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by juxtaposition of diagrams, e.g.

$$\left(\begin{array}{c} \text{Diagram A} \end{array} \right) \times \left(\begin{array}{c} \text{Diagram B} \end{array} \right) \mapsto \begin{array}{c} \text{Diagram C} \end{array}.$$

This concludes the definition of $\mathcal{U}(\mathfrak{gl}_n)$.

Note that for two 1-morphisms x and y in $\mathcal{U}(\mathfrak{gl}_n)$ the 2-hom-space $\text{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y)$ only contains 2-morphisms of degree zero and is therefore finite-dimensional. Following Khovanov and Lauda we introduce the graded 2-hom-space

$$\text{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y) = \bigoplus_{t \in \mathbb{Z}} \text{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x\{t\}, y),$$

which is infinite-dimensional. We also define the 2-category $\mathcal{U}(\mathfrak{gl}_n)^*$ which has the same objects and 1-morphisms as $\mathcal{U}(\mathfrak{gl}_n)$, but for two 1-morphisms x and y the vector space of 2-morphisms is defined by

$$(2.3.27) \quad \mathcal{U}(\mathfrak{gl}_n)^*(x, y) = \text{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x, y).$$

Note that $\mathcal{U}(\mathfrak{sl}_n)$ is defined just as $\mathcal{U}(\mathfrak{gl}_n)$, but labeling all the regions of the diagrams with \mathfrak{sl}_n -weights, i.e. elements of \mathbb{Z}^{n-1} . One also has to renormalize the signs of the left cups and caps, so that the bubble relations all become dependent on the \mathfrak{sl}_n -weights. For more details, see [41].

2.3.4. The q -Schur 2-algebra. The categorification of $S_q(n, n)$ is now obtained from $\mathcal{U}(\mathfrak{gl}_n)$ by taking a quotient.

Definition 2.3.10. The 2-category $\mathcal{S}(n, n)$ is the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n, n)$.

We remark that we only put real bubbles, whose interior has a label outside $\Lambda(n, n)$, equal to zero. To see what happens to a fake bubble, one first has to write it in terms of real bubbles with the opposite orientation using the infinite Grassmannian relation (2.3.18).

A main result of [41], given in Theorem 7.11 in that paper, is:

Theorem 2.3.11. *Let $K_0(\dot{\mathcal{S}}(n, n))$ denote the split Grothendieck group of the Karoubi envelope of $\mathcal{S}(n, n)$. The $\mathbb{Z}[q, q^{-1}]$ linear map*

$$\gamma_S: S_q^{\mathbb{Z}}(n, n) \rightarrow K_0(\dot{\mathcal{S}}(n, n)),$$

determined by

$$\gamma_S(E_{\underline{i}}1_{\lambda}) := [\mathcal{E}_{\underline{i}}1_{\lambda}]$$

is an isomorphism of algebras.

Recall also (Definition 4.1 in [41]) that there is an essentially surjective and full additive 2-functor

$$\Psi_{n,n}: \mathcal{U}(\mathfrak{sl}_n) \rightarrow \mathcal{S}(n, n),$$

whose precise definition is not relevant here. Up to signs related to cups and caps, it is obtained by mapping any string diagram to itself and applying $\phi_{n,n}$ to the labels of the regions. By convention, any diagram with a region labeled $*$ is taken to be zero. It is important to note that

$$K_0(\Psi_{n,n}): K_0(\dot{\mathcal{U}}(\mathfrak{sl}_n)) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}(q) \rightarrow K_0(\dot{\mathcal{S}}(n, n)) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}(q)$$

corresponds to the aforementioned surjective homomorphism

$$\psi_{n,n}: \dot{\mathcal{U}}(\mathfrak{sl}_n) \rightarrow S_q(n, n).$$

2.3.5. The cyclotomic KLR-algebras. In this subsection, we recall the definition of the cyclotomic KLR-algebras, due to Khovanov and Lauda [30] and, independently, to Rouquier [51]. We also recall two important results about them.

Fix $\nu \in \mathbb{Z}_{\leq 0}[I]$. Let $\text{Seq}(\nu)$ be the set of all sequences $\underline{i} = (-i_1, -i_2, \dots, -i_m)$, such that $i_k \in I$ for each k and $\nu_j = \#\{k \mid i_k = j\}$.

Definition 2.3.12. For any $\underline{i}, \underline{j} \in \text{Seq}(\nu)$ and any \mathfrak{gl}_n -weight $\lambda \in \mathbb{Z}^n$, let

$${}_{\underline{i}}R(\nu)_{\underline{j}} \subset \text{End}_{\mathcal{U}(\mathfrak{gl}_n)}(\mathcal{E}_{\underline{i}}1_{\lambda}, \mathcal{E}_{\underline{j}}1_{\lambda})$$

be the subalgebra containing only diagrams which are oriented downwards. So, only strands oriented downwards with dots and crossings are allowed. No strands oriented upwards, no cups and no caps. The relations in $\mathcal{U}(\mathfrak{gl}_n)$ involving only downward strands do not depend on λ . Therefore, the definition above makes sense. In [30], the authors do not label the regions of the diagrams.

Then $R(\nu)$ is defined as

$$R(\nu) := \bigoplus_{\underline{i}, \underline{j} \in \text{Seq}(\nu)} {}_{\underline{i}}R(\nu)_{\underline{j}}.$$

The ring R is defined as

$$R := \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R(\nu).$$

As remarked above, the definition of $R(\nu)$ does not depend on λ . However, when we use a particular λ , we will write $R(\nu)1_{\lambda}$.

Note that $R(\nu)$ is unital, whereas R has infinitely many idempotents.

Let $R(\nu)\text{-pmod}_{\text{gr}}$ be the category of graded finite-generated projective $R(\nu)$ -modules and define

$$R\text{-pmod}_{\text{gr}} := \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R(\nu)\text{-pmod}_{\text{gr}}.$$

In Proposition 3.18 in [30], Khovanov and Lauda showed that $R\text{-pmod}_{\text{gr}}$ categorifies the negative half of $\dot{\mathcal{U}}(\mathfrak{sl}_n)$ and $R(\nu)\text{-pmod}_{\text{gr}}$ categorifies the ν -root space.

We can now recall the definition of the *cyclotomic KLR-algebras*. The reader can find more details in [30] or [51], for example.

Definition 2.3.13. Choose a dominant $\dot{\mathcal{U}}(\mathfrak{gl}_n)$ -weight $\lambda \in \Lambda(n, n)^+$. Let $R(\nu; \lambda)$ be the quotient algebra of $R(\nu)1_\lambda$ by the ideal generated by all diagrams of the form

$$\begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \quad \dots \quad \downarrow \\ i_1 \quad i_2 \quad i_3 \quad \dots \quad i_m \end{array} \quad \begin{array}{c} \bar{\lambda}_m \\ \bullet \\ \lambda \end{array} \quad .$$

Define

$$R_\lambda := \bigoplus_{\nu \in \mathbb{Z}_{\leq 0}[I]} R(\nu; \lambda).$$

Recall that $\bar{\lambda}_m = \lambda_m - \lambda_{m+1}$, the m -th entry of the \mathfrak{sl}_n -weight corresponding to λ .

Note that we mod out by relations involving dots on the last strand, rather than the first strand as in [30]. This is to make the definition compatible with the other definitions in our paper.

It turns out that R_λ is a finite-dimensional unital algebra. Let $R_\lambda\text{-pmod}_{\text{gr}}$ be its category of finite-dimensional graded projective modules and let $K_0(R_\lambda)$ be the split Grothendieck group of that category.

There is a graded categorical action of $\mathcal{U}(\mathfrak{sl}_n)$ on $R_\lambda\text{-pmod}_{\text{gr}}$ and Brundan and Kleshchev [6] (see also [25], [38], [58] and [59]) proved a conjecture by Khovanov and Lauda:

Theorem 2.3.14. *We have*

$$K_0(R_\lambda) \cong V_\lambda^{\mathbb{Z}}$$

as $\dot{\mathcal{U}}^{\mathbb{Z}}(\mathfrak{sl}_n)$ -modules. Here $V_\lambda^{\mathbb{Z}}$ is the irreducible $\dot{\mathcal{U}}^{\mathbb{Z}}(\mathfrak{sl}_n)$ -module with highest weight $\bar{\lambda}$.

Rouquier's showed that, in a certain sense, R_λ is the universal categorification of V_λ . For a proof of the following result, see Lemma 5.4, Proposition 5.6 and Corollary 5.7 in [51].

Proposition 2.3.15. *Let \mathcal{V} be any additive idempotent complete category, which allows an integrable graded categorical action by $\mathcal{U}(\mathfrak{sl}_n)$ (for the precise definition see [51]). Suppose V_h is a highest weight object in \mathcal{V} , i.e an object that is killed by \mathcal{E}_{+i} , for all $i \in I$, and $\text{End}_{\mathcal{V}}(V_h) \cong \mathbb{C}$. Suppose also that any object in \mathcal{V} is a direct summand of XV_h , for some object $X \in \mathcal{U}(\mathfrak{sl}_n)$. Then there exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$ representations*

$$\Phi: R_\lambda\text{-pmod}_{\text{gr}} \rightarrow \mathcal{V}.$$

There are some subtle differences between Rouquier's approach to categorification and Khovanov and Lauda's. However, Proposition 2.3.15 holds in both setups, as already remarked by Webster in Section 1.4 in [59]. The proof of Proposition 2.3.15 consists of Rouquier's remarks in Section 5.1.2 and of the contents of his proofs of Lemma 5.4 and Proposition 5.6 in [51], which only rely on the assumptions in the statement of our Proposition 2.3.15 and the fact that \mathcal{E}_{+i} and \mathcal{E}_{-i} are biadjoint in $\mathcal{U}(\mathfrak{sl}_n)$, for any $i \in I$. The precise definition of the units and the counits, i.e. the cups and the caps, is not relevant for the validity of the proof. Note that we have included the hypothesis

$$\text{End}_{\mathcal{V}}(V_h) \cong \mathbb{C}$$

in Proposition 2.3.15, which is not one of Rouquier's assumptions. There are categorifications of V_λ without that property; see Conjecture 7.16 in [41] for example. However, in order to get a categorification which is really equivalent to $R_\lambda\text{-pmod}_{\text{gr}}$, i.e. with hom-spaces of the same graded dimension, one needs to add that assumption because it holds in the latter category.

We do not need the precise definition of Φ in this paper. In order to contain the length of this paper within reasonable boundaries, we will not explain it here.

3. THE \mathfrak{sl}_3 WEB ALGEBRA \mathcal{W}_S^c

For the rest of this section, let S be a fixed sign string of length n . We are going to define the *web algebra* \mathcal{W}_S^c .

Definition 3.0.16. (Web algebra) For $u, v \in B_S$, we define

$${}_u\mathcal{W}_v^c := \mathcal{F}^c(u^*v)\{n\},$$

where $\{n\}$ denotes a grading shift upwards in degree by n .

The *web algebra* \mathcal{W}_S^c is defined by

$$\mathcal{W}_S^c := \bigoplus_{u,v \in B_S} {}_u\mathcal{W}_v^c.$$

The multiplication on \mathcal{W}_S^c is defined by taking

$${}_u\mathcal{W}_{v_1}^c \otimes {}_{v_2}\mathcal{W}_w^c \rightarrow {}_u\mathcal{W}_w^c$$

to be zero, if $v_1 \neq v_2$, and by the map to be defined in Definition 3.0.18, if $v_1 = v_2 = v$.

Remark 3.0.17. In Proposition 3.0.21 we prove that the multiplication foam always has degree n , so the degree shift in the definition above makes \mathcal{W}_S^0 into a graded algebra and, for any $c \neq 0$, it makes \mathcal{W}_S^c into a filtered algebra.

Definition 3.0.18. (Multiplication of closed webs) The *multiplication*

$${}_u\mathcal{W}_v^c \otimes {}_v\mathcal{W}_w^c \rightarrow {}_u\mathcal{W}_w^c$$

is induced by the *multiplication foam*

$$m_{u,v,w} : u^*vv^*w \xrightarrow{Id_u * m_v Id_w} u^*w,$$


where $m_v : vv^* \rightarrow \text{Vert}_n$, with Vert_n being the web of n parallel oriented vertical line segments, is defined by the following inductive algorithm:

- (1) Express v using the growth algorithm, label each level of the growth algorithm starting from zero. Then form vv^* .
- (2) At the k th level in the growth algorithm, *resolve* the corresponding pair of arc, H or Y-rules in v and v^* by applying the foams:

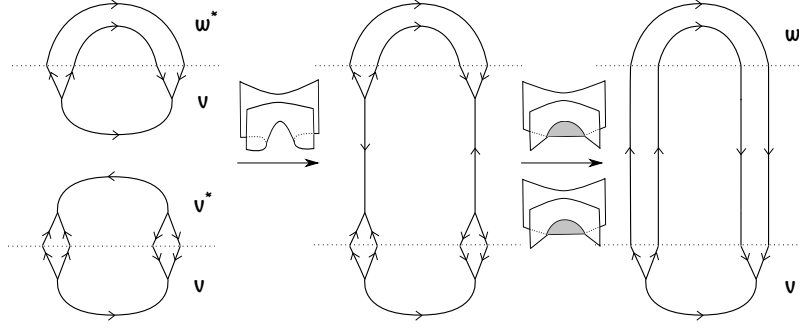
$$(3.0.28) \quad \begin{array}{c} \text{Diagram 1: A crossing of two lines with a foam on top.} \end{array} \rightarrow \begin{array}{c} \text{Diagram 2: Two parallel vertical lines with a foam on top.} \end{array} \quad \begin{array}{c} \text{Diagram 3: A Y-junction with a foam on top.} \end{array} \rightarrow \begin{array}{c} \text{Diagram 4: Two parallel vertical lines with a foam on top.} \end{array} \quad \begin{array}{c} \text{Diagram 5: A square with a foam on top.} \end{array} \rightarrow \begin{array}{c} \text{Diagram 6: Two parallel vertical lines with a foam on top.} \end{array}$$

Note that at the last level in the growth algorithm of v , only pairs of arcs are present.

Example 3.0.19. Let w and v be the following webs

(3.0.29) 

the multiplication foam $m_{w,v,v}$ is given by the following steps

(3.0.30) 

Proposition 3.0.20. The foam m_v in Def. 3.0.18 only depends on the isotopy type of v .

Proof. We have to show that m_v is independent of the way v is expressed using the growth algorithm (Def. 2.1.1). Let G_1 and G_2 be two different expressions of v using the growth algorithm. We have to compare G_1 and G_2 walking backwards in the growth algorithm. Note that we only have to worry about two consecutive steps in the same region of v . Reordering steps in “distant” regions of v corresponds to an isotopy which simply alters the height function on m_v . With these observations, the only possible remaining difference between the last two steps in G_1 and G_2 is the following:

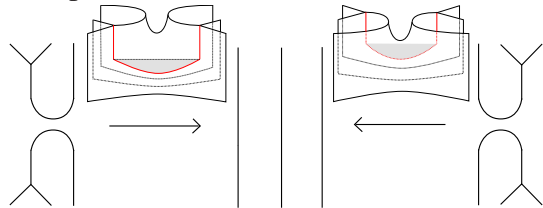
(3.0.31) 

If the last two steps in G_1 and G_2 are equal, we have to go further back in the growth algorithm. Besides two-step differences of the same sort as above, we can encounter another one of the following sort:

(3.0.32) 

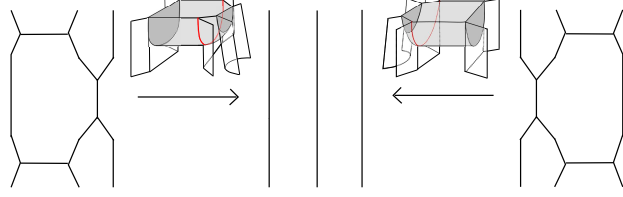
We have to check that the above two-step differences in G_1 and G_2 correspond to equivalent foams. In the first case, the foams in the multiplication algorithm are given by

FIGURE 3. A possible local difference between m_{G_1} and m_{G_2} .



In the second case, we get

FIGURE 4. The other possible local difference between m_{G_1} and m_{G_2} .



The two foams in Figure 3 are isotopic - one foam can be produced from the other by sliding the red singular arc over the saddle:

$$(3.0.33) \quad \text{[Diagram showing isotopy of foams by sliding a red singular arc over a saddle point]} = \text{[Diagram showing the result of the isotopy]}$$

The two foams in Figure 4 are also isotopic - one foam can be produced from the other by moving the red singular arc to the right or to the left:

$$(3.0.34) \quad \text{[Diagram showing isotopy of foams by moving a red singular arc left or right]} = \text{[Diagram showing the result of the isotopy]}$$

The cases above are the only possible ones, so their verification provides the proof. \square

Proposition 3.0.21. *The foam m_v has q -grading n .*

Proof. We proceed by backward induction on the level of the growth algorithm expressing v . At the final level of the growth algorithm, the only possible rule is the arc rule. Resolving a corresponding pair of arcs in v and v^* results in two new vertical strands and is obtained by a saddle point cobordism, which has q -grading 2.

Let n_k be the number of vertical strands and m_v^k be the foam after resolving the last k rules in the growth algorithm of v . Suppose that n_k is equal to the q -degree of m_v^k . In the next step of the multiplication we can have:

- (1) The resolution of a pair of arc rules. In this case we have $n_{k+1} = n_k + 2$ and m_v^{k+1} is obtained from m_v^k by adding a saddle, which adds 2 to the q -grading.
- (2) The resolution of a pair of Y rules. In this case we have $n_{k+1} = n_k + 1$ and m_v^{k+1} is obtained from m_v^k by adding an unzip, which adds 1 to the q -grading.
- (3) The resolution of a pair of H rules. In this case we have $n_{k+1} = n_k$ and m_v^{k+1} is obtained from m_v^k by adding a square foam, which adds 0 to the q -grading.

\square

There is a useful alternative definition of \mathcal{W}_S^c , which we give below. As a service to the reader, we state it as a lemma and prove that it really is equivalent to our definition above. Both definitions have their advantages and disadvantages, so it is worthwhile to catalogue both in this paper.

Lemma 3.0.22. *For any $c \in \mathbb{C}$ and any $u, v \in B_S$, we have a grading preserving isomorphism*

$$\mathbf{Foam}_3^c(u, v) \cong {}_u\mathcal{W}_v^c.$$

Using this isomorphism, the multiplication

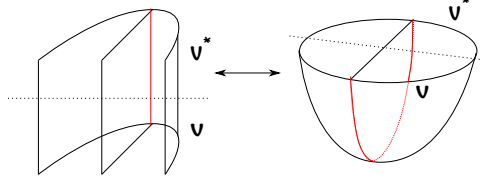
$${}_u\mathcal{W}_v^c \otimes {}_{v'}\mathcal{W}_w^c \rightarrow {}_u\mathcal{W}_w^c$$

corresponds to the composition

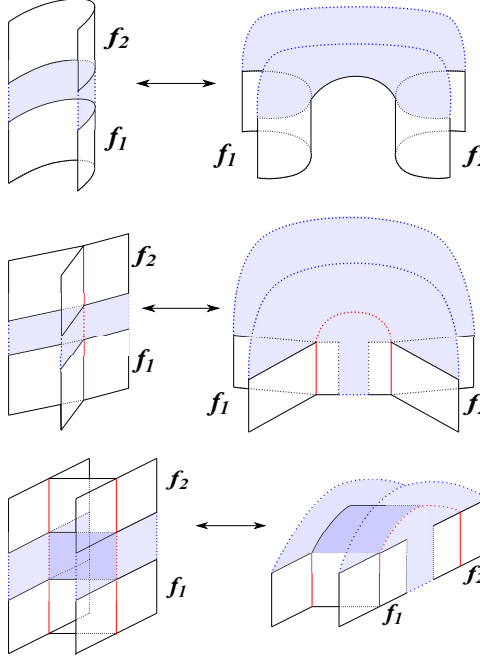
$$\mathbf{Foam}_3^c(u, v) \otimes \mathbf{Foam}_3^c(v', w) \rightarrow \mathbf{Foam}_3^c(u, w),$$

if $v = v'$, and is zero otherwise.

Proof. The isomorphism of the first claim is sketched in the following figure:



The proof of the second claim follows from analyzing what the isomorphism does to the resolution of a pair of arc, Y or H-rules in the multiplication foam. This is done below.



□

Note that Lemma 3.0.22 implies that \mathcal{W}_S^c is associative and unital, something that is not immediately clear from Definition 3.0.16. For any $u \in B_S$, the identity $1_u \in \mathbf{Foam}_3^c(u, u)$ defines an idempotent. We have

$$1 = \sum_{u \in B_S} 1_u \in \mathcal{W}_S^c.$$

Alternatively, one can see \mathcal{W}_S^c as a category whose objects are the elements in B_S such that the module of morphisms between $u \in B_S$ and $v \in B_S$ is given by $\mathbf{Foam}_3^c(u, v)$. In this paper we will mostly see \mathcal{W}_S^c as an algebra, but will sometimes refer to the category point of view.

In this paper, we will study \mathcal{W}_S^c for two special values of $c \in \mathbb{C}$.

Definition 3.0.23. Let K_S and G_S be the complex algebras obtained from \mathcal{W}_S^c by setting $c = 0$ and $c = 1$, respectively. We call them *Khovanov's web algebra* and *Gornik's web algebra*, respectively, to distinguish them throughout the paper.

Note that G_S is a filtered algebra. Its associated graded algebra is K_S . By Lemma 3.0.22, both K_S and G_S are finite-dimensional, unital, associative algebras. They also have similar decompositions:

$$K_S = \bigoplus_{u,v \in B_S} {}_u K_v, \quad G_S = \bigoplus_{u,v \in B_S} {}_u G_v.$$

We now recall the definition of complex graded and filtered Frobenius algebras. Let A be a finite-dimensional graded complex algebra and let $\text{hom}_{\mathbb{C}}(A, \mathbb{C})$ be the complex vector space of grading preserving maps. The *dual* of A is defined as

$$A^\vee := \bigoplus_{n \in \mathbb{Z}} \text{hom}_R(A, \mathbb{C}\{n\}),$$

where $\{n\}$ denotes an upward degree shift of size n . Note that A^\vee is also a graded algebra, such that

$$(3.0.35) \quad (A^\vee)_i = (A_{-i})^\vee,$$

for any $i \in \mathbb{Z}$. Then A is called a *graded symmetric Frobenius algebra of Gorenstein parameter ℓ* , if there exists an isomorphism of graded (A, A) -bimodules

$$A^\vee \cong A\{-\ell\}.$$

If A is a complex finite-dimensional filtered algebra, let $\text{hom}_{\mathbb{C}}(A, \mathbb{C})$ be the complex vector space of filtration preserving maps. The *dual* of A is defined as

$$A^\vee := \bigoplus_{n \in \mathbb{Z}} \text{hom}_R(A, \mathbb{C}\{n\}),$$

where $\{n\}$ denotes an upward suspension of size n . Note that A^\vee is also a filtered algebra, such that

$$(3.0.36) \quad (A^\vee)_i = (A_{-i})^\vee,$$

for any $i \in \mathbb{Z}$. Then A is called a *filtered symmetric Frobenius algebra of Gorenstein parameter ℓ* , if there exists an isomorphism of filtered (A, A) -bimodules

$$A^\vee \cong A\{-\ell\}.$$

For more information on graded Frobenius algebras, see [57] and the references therein, for example. We do not have a good reference for filtered Frobenius algebras, but it is a straightforward generalization of the graded case. We will explain some basic results on the character theory of filtered and graded symmetric Frobenius algebras in Section 5.

Theorem 3.0.24. *For any sign string S of length n , the algebra K_S is a graded symmetric Frobenius algebra and G_S is a filtered symmetric Frobenius algebra, both of Gorenstein parameter $2n$.*

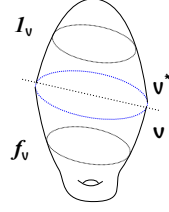
Proof. First, let $c = 0$. We take, by definition, the trace form

$$\text{tr}: K_S \rightarrow \mathbb{C}$$

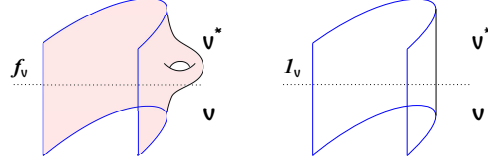
to be zero on ${}_u K_v$, when $u \neq v \in B_S$. For any $v \in B_S$, we define

$$\text{tr}: {}_v K_v \rightarrow \mathbb{C}$$

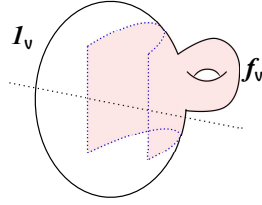
by closing any foam f_v with 1_v , e.g.



Equivalently, in $\mathbf{Foam}_3^0(v, v)$, closing f_v by 1_v ,

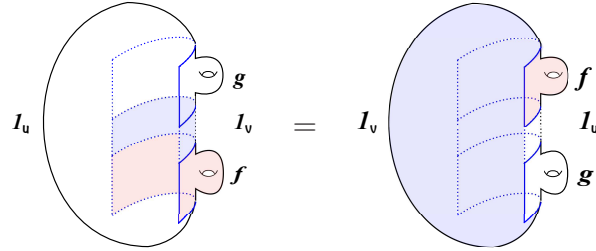


gives



The fact that the trace form is non-degenerate follows immediately from the closure relation in Subsection 2.2.

The fact that $\text{tr}(gf) = \text{tr}(fg)$ holds follows from sliding f around the closure until it appears on the other side of g , e.g.



Note that a closed foam can only have non-zero evaluation if it has degree zero. Therefore, for any $u \in B_S$ and any two homogeneous elements $f \in \mathcal{F}^0(u^*v)$ and $g \in \mathcal{F}^0(v^*u)$, we have $\text{tr}(fg) \neq 0$ unless $\deg(f) = -\deg(g)$. By the shift in

$${}_u K_v = \mathcal{F}^0(u^*v)\{n\}$$

and by (3.0.35), this implies that the non-degenerate trace form on K_S gives rise to a graded (K_S, K_S) -bimodule isomorphism

$$(3.0.37) \quad K_S^\vee \cong K_S\{-2n\}.$$

Now, let $c = 1$. Then the construction above also gives a non-degenerate bilinear form on G_S . Moreover, it induces a filtration preserving bijective \mathbb{C} -linear map of filtered (G_S, G_S) -bimodules

$$(3.0.38) \quad G_S\{-2n\} \rightarrow G_S^\vee.$$

The associated graded map is precisely the isomorphism in (3.0.37). By Proposition A.0.25, this implies that the map in (3.0.38) is a strict isomorphism of filtered (G_S, G_S) -bimodules. \square

We now explain some of Gornik's results, which are relevant for G_S . Recall that $R_{u^*v}^1$ is the commutative ring associated to u^*v , generated by the edge variables of u^*v and mod out by the ideal, which, for each trivalent vertex in u^*v , is generated by the relations

$$(3.0.39) \quad x_1 + x_2 + x_3 = 0, \quad x_1x_2 + x_1x_3 + x_2x_3 = 0, \quad x_1x_2x_3 = 1,$$

where x_1, x_2 and x_3 are the edge variables around the vertex. The algebra $R_{u^*v}^1$ acts on ${}_uG_v$ in such a way that each edge variable corresponds to adding a dot on the incident facet. See [21, 27, 42] for the precise definition and more details.

In what follows, 3-colorings will always be assumed to be admissible and we therefore omit the adjective. Theorem 3 in [21] proves the following:

Theorem 3.0.25 (Gornik). *There is a complete set of orthogonal idempotents $e_T \in R_{u^*v}^1$, indexed by the 3-colorings T of u^*v . The number of 3-colorings of u^*v is exactly equal to $\dim_q({}_uG_v)$.*

*These idempotents are not filtration preserving, but as an $R_{u^*v}^1$ -module (i.e. forgetting the filtration on ${}_uG_v$ and its left ${}_uG_u$ and right ${}_vG_v$ -module structures) we have*

$${}_uG_v \cong \bigoplus_T \mathbb{C}e_T.$$

Let us have a closer look at Gornik's idempotents. First of all, in the proof of Theorem 3 in [21] Gornik notes that for any edge i and any 3-coloring T of u^*v , we have

$$(3.0.40) \quad x_i e_T = \zeta^{T_i} e_T \in R_{u^*v}^1,$$

where ζ is a primitive third root of unity, x_i is the edge variable and T_i the color of the edge (see (4) in [42] for this result in the context of foams).

Furthermore, a 3-coloring of u^*v actually corresponds to a pair of 3-colorings of u and v^* that match at the boundary. Of course, there is a bijective correspondence between 3-colorings of u and v^* , so we see that a 3-coloring of u^*v corresponds to a matching pair of 3-colorings of u and v .

Recall that ${}_uG_v$ is a left ${}_uG_u$ -module and a right ${}_vG_v$ -module. Let T_1 and T_2 be a pair of matching 3-colorings of u and v , respectively, which together give a 3-coloring T of u^*v . Then the action of e_T on any $f: u \rightarrow v$ can be written as

$$e_{T_1} f e_{T_2}.$$

To show that this notation really makes sense, define *Gornik's symmetric idempotent* associated to T_1 as

$$e_{u, T_1} := e_{T_1} 1_u e_{T_1}.$$

So we let the Gornik idempotent associated to the symmetric 3-coloring of u^*u , given by T_1 both on u and u^* , act on 1_u . Then we have

$$e_{T_1} f e_{T_2} = e_{u, T_1} f e_{v, T_2},$$

where on the right-hand side we really mean composition.

We immediately see that

$$e_{T_1} 1_u e_{T_2} = 0 \Leftrightarrow T_1 \neq T_2$$

and

$$e_{T_1} 1_u e_{T_1} e_{T_2} 1_u e_{T_2} = \delta_{1,2} e_{T_1} 1_u e_{T_1} \quad \text{and} \quad \sum_T e_T 1_u e_T = 1_u,$$

where the sum is over all 3-colorings of u . This shows that the $e_{u,T}$, for all 3-colorings T of a given $u \in B_S$, are orthogonal idempotents in ${}_uG_u$. It also implies that

$$e_{T_1} 1_u e_{T_1} = e_{T_1} 1_u = 1_u e_{T_1},$$

so it is enough to label just the source or just the target of 1_u . For this purpose, we define R_u^1 to be “half” of $R_{u^*u}^1$, i.e. the subring which is only generated by the edge variables of u . To be precise, we have

$$R_{u^*u}^1 \cong R_u^1 \otimes_S R_u^1,$$

where \otimes_S indicates that we impose the relation $x \otimes 1 = 1 \otimes x$, for any x corresponding to a boundary edge of u .

If u has no closed cycles, then all the 3-colorings of u^*u are symmetric, because they are completely determined by the colors on the boundary of u . In that case

$$e_T \mapsto e_{u,T}$$

defines an isomorphism of algebras $R_u^1 \cong {}_uG_u$. In particular, ${}_uG_u$ is commutative. This is not true in general, but we can prove the following:

Lemma 3.0.26. *For any $u \in B_S$, the map*

$$x \mapsto x 1_u$$

defines a strict embedding of filtered R_u^1 -modules

$$\iota: R_u^1 \rightarrow {}_uG_u.$$

In particular, we see that $(R_u^1)_0 \cong \text{Im}(\iota)_0 \cong \mathbb{C}1_u$.

Proof. The map is clearly a homomorphism of filtered algebras.

The relations (Dot Migration) correspond precisely to the relations in R_u^1 , because the only singular edges in 1_u are the ones corresponding to the trivalent vertices of u . This shows that it is a strict embedding. \square

For any $u \in B_S$, we define the graded ring

$$R_u^0 := E(R_u^1).$$

This ring is the one which appears in Khovanov’s original paper [27]. In R_u^0 we have the relations

$$(3.0.41) \quad x_1 + x_2 + x_3 = 0, \quad x_1x_2 + x_1x_3 + x_2x_3 = 0, \quad x_1x_2x_3 = 0.$$

The reader should compare them to (3.0.39).

There are no analogues of the Gornik idempotents in R_u^0 , but we do have an analogue of Lemma 3.0.26.

Lemma 3.0.27. *For any $u \in B_S$, the map*

$$x \mapsto x 1_u$$

defines an embedding of graded R_u^0 -modules

$$E(\iota): R_u^0 \rightarrow {}_uK_u.$$

In particular, we see that $(R_u^0)_0 \cong \text{Im}(E(\iota))_0 \cong \mathbb{C}1_u$.

Another interesting consequence of Theorem 3.0.25 is the following.

Proposition 3.0.28. *As a complex algebra, i.e. without taking the filtration into account, G_S is semisimple.*

Proof. For any $u \in B_S$ and any 3-coloring T of u , define the projective G_S -module

$$P_{u,T} := (G_S)e_{u,T},$$

where $e_{u,T}$ is Gornik's symmetric idempotent in G_S defined above. Theorem 3.0.25 and our subsequent analysis of Gornik's idempotents show that the $P_{u,T}$ form a complete set of indecomposable projective G_S -modules. Furthermore, we have

$$\mathrm{Hom}_{G_S}(P_{u,T}, P_{v,T'}) \cong e_{u,T}(G_S)e_{v,T'} \cong \begin{cases} \mathbb{C}, & \text{if } T \text{ and } T' \text{ match at } S, \\ \{0\}, & \text{else.} \end{cases}$$

This shows that $P_{u,T} \cong P_{v,T'}$ if and only if T and T' match at the common boundary. It also shows that if $P_{u,T} \not\cong P_{v,T'}$, then

$$\mathrm{Hom}_{G_S}(P_{u,T}, P_{v,T'}) = \mathrm{Hom}_{G_S}(P_{v,T'}, P_{u,T}) = \{0\}.$$

Finally, it shows that each $P_{u,T}$ has only one composition factor, i.e. $P_{u,T}$ is irreducible.

It is well-known that this implies that G_S is semisimple; see Proposition 1.8.5 in [3] for example. \square

By Proposition 3.0.28, it is clear that for each $u \in B_S$ and each coloring T of u , the corresponding block in G_S is isomorphic to $\mathrm{End}(P_{u,T})$. In Section 4, we will determine the central idempotents of G_S .

4. THE CENTER OF THE WEB ALGEBRA AND THE COHOMOLOGY RING OF THE SPALTENSTEIN VARIETY

For the rest of this section, choose arbitrary but fixed non-negative integers $n \geq 2$ and $k \leq n$, such that $d := 3k \geq n$. Let

$$\Lambda(n, d) := \left\{ \mu \in \mathbb{N}^n \mid \sum_{i=1}^n \mu_i = d \right\}$$

be the set of *compositions* of d of length n . By $\Lambda^+(n, d) \subset \Lambda(n, d)$ we denote the subset of *partitions*, i.e. all $\mu \in \Lambda(n, d)$ such that

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0.$$

Also for the rest of this section, choose an arbitrary but fixed sign string S of length n . We associate to S a unique element $\mu = \mu_S \in \Lambda(n, d)$, such that

$$\mu_i = \begin{cases} 1, & \text{if } s_i = +, \\ 2, & \text{if } s_i = -. \end{cases}$$

Let $\Lambda(n, d)_{1,2} \subset \Lambda(n, d)$ be the subset of compositions whose entries are all 1 or 2. For any sign string S , we have $\mu_S \in \Lambda(n, d)_{1,2}$.

Let $\lambda = (3^k) \in \Lambda(n, d)$. Let Col_μ^λ be the set of column strict tableaux of shape λ and type μ , both of length n . It is well-known that there is a bijection between Col_μ^λ and the tensor basis of

$$V_\mu := V_{\mu_1} \otimes \dots \otimes V_{\mu_n},$$

where $V_1 = V_+$ and $V_2 = V_1 \wedge V_1 \cong V_-$ (see Section 3 in [46], for example). However, we are interested in tensors as summands in the decomposition of elements in B_S . Therefore, we prove Proposition 4.1.2 in Subsection 4.1. The reader, who is not interested in the details of the proof of this proposition, can choose to skip this subsection at a first reading and just read the statement of the proposition.

4.1. Tableaux and flows. Let p_S be the number of positive entries and n_S the number of negative entries of S . By definition, we have that $d = p_S + 2n_S$. The key idea in this subsection is to reduce all proofs to the case where $n_S = 0$.

Definition 4.1.1. Fix any state string J of length n , we define a new state string \hat{J} of length d by the following algorithm:

- (1) Let ${}_0\hat{J}$ be the empty string.
- (2) For $1 \leq i \leq n$, let ${}_i\hat{J}$ be the result of concatenating j_i to ${}_{i-1}\hat{J}$ if $\mu_i = 1$. If $\mu_i = 2$ then
 - (a) concatenate $(1, 0)$ to ${}_{i-1}\hat{J}$ if $j_i = 1$,
 - (b) concatenate $(0, -1)$ to ${}_{i-1}\hat{J}$ if $j_i = -1$,
 - (c) concatenate $(1, -1)$ to ${}_{i-1}\hat{J}$ if $j_i = 0$.

We set $\hat{J} = {}_n\hat{J}$. Lastly, for any $c \in \{-1, 0, 1\}$, we define \hat{J}^c to be the number of entries in \hat{J} that is equal to c .

Proposition 4.1.2. *There is a bijection between Col_μ^λ and the set of state strings J such that there exists a $w \in B_S$ and a flow f on w which extends J .*

The proof of Proposition 4.1.2 follows directly from Lemmas 4.1.3 and 4.1.4.

Lemma 4.1.3. *There is a bijection between Col_μ^λ and state strings J of length n such that*

$$(4.1.1) \quad \hat{J}^{-1} = \hat{J}^0 = \hat{J}^1.$$

where the \hat{J}^c are as defined in Definition 4.1.1.

Proof. Given a state string J satisfying (4.1.1), we first give an algorithm to build a 3-column tableau Y_J , filled with integers from 1 to n . Afterwards, we show that Y_J has shape λ .

Begin by labeling the three columns with 1, 0 and -1 , reading from left to right. We are going to build up Y_J from top to bottom. Start by taking Y_J to be the empty tableau. Then, from $i = 1$ to $i = n$, do the following:

- (1) If $\mu_i = 1$, add one box labeled i to column j_i in Y_J .
- (2) If $\mu_i = 2$, add two boxes labeled i to columns c_1 and c_2 , such that $c_1 \neq c_2$ and $c_1 + c_2 = j_i$.

We have to show that Y_J belongs to Col_μ^λ . Since the algorithm builds up from top to bottom, Y_J is strictly column increasing. To see that Y_J has shape λ , we need to show that every row in Y_J has three entries. Observe that the number of filled boxes in column c of Y_J is exactly equal to \hat{J}^c . Since we have assumed condition (4.1.1), all three columns have the same length, therefore every row in Y_J must have exactly three entries.

Conversely, let $T \in \text{Col}_\mu^\lambda$. We define a state string J as follows:

$$j_i = \sum_{i \text{ appears in column } c} c.$$

Since μ corresponds to a sign string and T is column strict, we see that, for each $1 \leq i \leq n$, i can appear at most twice in T but never twice in the same column. Thus, $j_i \in \{-1, 0, 1\}$, i.e. J is

a state string. It follows from the definition of \hat{J} that \hat{J}^c is equal to the length of column c of T . Since T is of shape λ , the number of boxes in each column is the same. Hence, condition (4.1.1) holds for J .

It is straightforward to check that the above two constructions are inverse to each other and therefore determine a bijection. \square

Lemma 4.1.4. *A state string J corresponds to the boundary state of a flow on a web $w \in B_S$ if and only if condition (4.1.1) holds for J .*

Proof. Let $w \in B_S$ be equipped with a flow with boundary state string J . We are going to show that J satisfies condition (4.1.1) by induction on n . For $n = 2$, w can only be an arc. In this case it is simple to check that all flows on w have corresponding boundary state strings satisfying condition (4.1.1).

For $n > 2$, we express w using the growth algorithm in an arbitrary, but fixed way, with the restriction that only one rule is applied per level. Let ${}_k J$ denote the boundary state string at the beginning of the k -th level in the growth algorithm and ${}_k \hat{J}$ the associated string as in Definition 4.1.1. Similarly, let ${}_k \mu$ denote the composition corresponding to the sign string at the k -th level. Let us compare ${}_{k+1} J$ and ${}_k J$. They can only differ in the following ways:

- (1) In case an arc-rule is applied at the k -th level, ${}_k J$ can be obtained from ${}_{k+1} J$ by inserting the substring $(1, -1)$, $(0, 0)$ or $(-1, 1)$ between the i -th and $i + 1$ -th entries in ${}_{k+1} J$. ${}_k \mu$ can be obtained from ${}_{k+1} \mu$ by inserting the substring $(1, 2)$ or $(2, 1)$ between the i -th and $i + 1$ -th entries in ${}_{k+1} \mu$:

$$(4.1.2) \quad \begin{array}{c} 1 \quad -1 \quad 0 \quad -1 \quad 1 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}.$$

- (2) In case a Y-rule is applied, ${}_k J$ can be obtained from ${}_{k+1} J$ by replacing the i -th entry in ${}_{k+1} J$ with a length two substring whose sum is equal to the i -th entry. ${}_k \mu$ can be obtained from ${}_{k+1} \mu$ by replacing the i -th entry in ${}_{k+1} \mu$ with the substring $(3 - {}_{k+1} \mu_i, 3 - {}_{k+1} \mu_i)$:

$$(4.1.3) \quad \begin{array}{c} 1 \quad -1 \quad 1 \quad 0 \quad 0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0 \quad -1 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}.$$

- (3) In case an H-rule is applied, ${}_k \mu$ can be obtained from ${}_{k+1} \mu$ by replacing a substring $(1, 2)$ or $(2, 1)$, at the i -th and $(i + 1)$ -th position in ${}_{k+1} \mu$, with $(3 - {}_{k+1} \mu_i, 3 - {}_{k+1} \mu_{i+1})$. ${}_k J$ can be obtained from ${}_{k+1} J$ by replacing a substring of length two in ${}_{k+1} J$ at the i -th and $(i + 1)$ -th position according to the schema:

$$(4.1.4) \quad \begin{array}{c} 1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 1 \quad 0 \quad -1 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

$$(4.1.5) \quad \begin{array}{c} 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1 \quad 1 \quad -1 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}.$$

It is straightforward to check that ${}_{k+1} J$ satisfies condition (4.1.1), with composition ${}_{k+1} \mu$, if and only if ${}_k J$ does, with composition ${}_k \mu$. Take, for example, an instance where a Y-rule is applied; suppose also that the i -th entry in ${}_{k+1} \mu$ is 2 and the i -th entry in ${}_{k+1} J$ is 0. Thus, the i -th entry in ${}_{k+1} J$ contributes a pair $(1, -1)$ to ${}_{k+1} \hat{J}$. By (4.1.3), ${}_k J$ is obtained from ${}_{k+1} J$ by replacing the i -th

entry in ${}_{k+1}J$ with $(1, -1)$ and the i -th entry in ${}_{k+1}\mu$ with $(1, 1)$. We see that ${}_k\hat{J}$ is in fact exactly equal to ${}_{k+1}\hat{J}$. Therefore ${}_{k+1}\hat{J}$ satisfies condition (4.1.1) if and only if ${}_k\hat{J}$ does. Similar analysis apply to all cases in (4.1.2), (4.1.3) and (4.1.4).

Let k be the first level in the growth algorithm of w where a Y or an arc-rule is applied. From the $(k + 1)$ -th level down we have a non-elliptic web w' with flow, whose boundary state string ${}_{k+1}J$ and composition ${}_{k+1}\mu$ both have length less than n . Thus, by our induction hypothesis, ${}_{k+1}J$, with composition ${}_{k+1}\mu$, satisfies condition (4.1.1).

By the above argument, then ${}_iJ$ also satisfy condition (4.1.1), for any $0 \leq i \leq k$. In particular, $J = {}_0J$ satisfies that condition, which is what we had to prove.

Conversely, let J satisfy condition (4.1.1), with composition μ . We show, by induction on n , that there is a $w \in B_S$ with flow whose boundary state string is exactly J . More specifically, we first construct a $w \in W_S$ and then show that w is non-elliptic, i.e. $w \in B_S$.

For $n = 2$, then w must be an arc. It is simple to check that if J satisfies condition (4.1.1), J is the boundary state of a flow on an arc.

For $n > 2$, suppose it is possible to apply an arc or Y-rule to the pair μ and J , depicted in (4.1.2) and (4.1.3). Then we obtain a new pair μ' and J' with length less than n . Thus, by induction, there exist a web $w' \in W_{S'}$ and flow extending J' . Gluing the arc or Y on top of w' results in a web $w \in W_S$ with a flow extending J .

Suppose, then, that it is not possible to apply an arc or Y-rule to μ and J . This means that one of the following must hold:

- (1) μ does not contain a substring of type $(1, 2)$ or $(2, 1)$ and $J = (1, \dots, 1)$, $J = (-1, \dots, -1)$ or $J = (0, \dots, 0)$.
- (2) μ contains at least one substring of the form $(1, 2)$ or $(2, 1)$. For every substring in μ of the form $(1, 2)$ or $(2, 1)$, the corresponding substring in J is $(\pm 1, \pm 1)$, $(0, 1)$ or $(1, 0)$. For every substring in μ of the form $(1, 1)$ or $(2, 2)$, the corresponding substring in J is $(1, 1)$, $(-1, -1)$ or $(0, 0)$.

Case 1 contradicts the assumption that J satisfies condition (4.1.1).

Case 2 contains several subcases, each of which contains details which are slightly different. However, the general idea is the same for all of them and is very simple: apply H-moves until you can apply an arc or a Y-rule and finish the proof by induction.

We first suppose, without loss of generality, that μ contains a substring $(\mu_i, \mu_{i+1}) = (1, 2)$ and that the corresponding substring in J is $(j_i, j_{i+1}) = (1, 1)$ (the subcase for $(j_i, j_{i+1}) = (-1, -1)$ is analogous). We see that $(1, 1)$ in J contributes a substring $(1, 0, 1)$ to \hat{J} . Thus, our assumption that \hat{J} satisfies condition (4.1.1) implies that \hat{J} contains at least one more entry equal to -1 . This means that for some $r \neq i, i + 1$, $1 \leq r \leq n$, one of the following is true:

- (a) $j_r = -1$, $\mu_r = 1$, denoted for brevity by

$$(4.1.6) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \end{array} \dots \begin{array}{c} \uparrow -1 \end{array} \dots$$

(b) $j_r = -1, \mu_r = 2$, denoted

$$(4.1.7) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \downarrow -1 \end{array} \dots$$

(c) $j_r = 0, \mu_r = 2$, denoted

$$(4.1.8) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \downarrow 0 \end{array} \dots$$

Without loss of generality, let us assume $i+1 < r$. Consider subcases (a) and (b). If $j_m \neq 0$ for all $i+1 < m < r$, then it is possible to apply an arc or Y-move to J and μ , contrary to our assumption in case 2. Thus, in all three scenarios above it suffices to analyze the following two configurations:

$$(4.1.9) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \uparrow 0 \end{array} \dots \qquad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \downarrow 0 \end{array} \dots$$

Let $i+1 < r \leq n$ be smallest integer where $j_r = 0$. We must have that $\mu_{r-1} = 3 - \mu_r$ and $j_{r-1} = \pm 1$. For any other values of μ_{r-1} and j_{r-1} we would be able to apply an arc or a Y-move, contradicting our assumptions for case 2. In both situations, we can apply an H-rule to the substrings (j_{r-1}, j_r) and (μ_{r-1}, μ_r) :

$$(4.1.10) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \uparrow \pm 1 \\ \downarrow 0 \end{array} \dots \qquad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 1 \\ \downarrow \pm 1 \\ \uparrow 0 \end{array} \dots$$

This results in new sign and state strings, each with length n satisfying condition (4.1.1). The application of the H-rule in (4.1.10) moves the zero at the r -th position to the $r-1$ position. Either we can now apply an arc or Y-rule to the new strings or by repeatedly applying an H-rule in the manner of (4.1.10), we obtain one of the following pairs:

$$(4.1.11) \quad \dots \begin{array}{c} \downarrow 1 \\ \downarrow 0 \end{array} \dots \qquad \dots \begin{array}{c} \uparrow 1 \\ \uparrow 0 \end{array} \dots$$

To either of the above diagrams we can apply a Y-rule, after which we can use induction.

To complete our analysis of case 2, now suppose, without loss of generality, that μ contains a substring $(\mu_i, \mu_{i+1}) = (1, 2)$ and that the corresponding substring in J is $(j_i, j_{i+1}) = (1, 0)$ (the subcases for $(0, \pm 1)$ or $(-1, 0)$ are analogous).

We see that $(1, 0)$ in J contributes a substring $(1, 1, -1)$ to \hat{J} . Thus, our assumption that \hat{J} satisfies condition (4.1.1) implies that \hat{J} contains at least one more entry equal to -1 . This means that for some r , with $1 \leq r \leq n$, one of the following is true:

(a) $j_r = -1, \mu_r = 1$, denoted for brevity by

$$(4.1.12) \quad \dots \begin{array}{c} \uparrow 1 \\ \downarrow 0 \\ \uparrow -1 \end{array} \dots$$

(b) $j_r = -1, \mu_r = 2$, denoted

$$(4.1.13) \quad \begin{array}{ccccc} & 1 & 0 & -1 & \\ \dots & \uparrow & \downarrow & \downarrow & \dots \end{array}$$

(c) $j_r = 0, \mu_r = 2$, denoted

$$(4.1.14) \quad \begin{array}{ccccc} & 1 & 0 & 0 & \\ \dots & \uparrow & \downarrow & \downarrow & \dots \end{array}$$

For subcases (a) and (b), if $\mu_{i+2} = 1$ and $j_{i+2} = -1$, we may apply an H-rule to (μ_{i+1}, μ_{i+2}) , (j_{i+1}, j_{i+2}) to obtain a new pair μ' and J' . Subsequently we can apply a Y-rule to the i -th and $(i+1)$ -th entries of μ' and J' :

$$(4.1.15) \quad \begin{array}{ccccc} & 1 & 0 & -1 & \\ \dots & \uparrow & \downarrow & \downarrow & \dots \end{array}$$

After applying the Y-rule, we can use induction.

Otherwise, we can show, just as before, that all three scenarios above reduce to an analysis of the following two configurations:

$$(4.1.16) \quad \begin{array}{ccccc} & 1 & 0 & 0 & \\ \dots & \uparrow & \downarrow & \uparrow & \dots \end{array} \quad \begin{array}{ccccc} & 1 & 0 & 0 & \\ \dots & \uparrow & \downarrow & \downarrow & \dots \end{array}$$

That is, we can assume that μ contains a substring $(\mu_i, \mu_{i+1}) = (1, 2)$ with the corresponding substring in J being $(j_i, j_{i+1}) = (1, 0)$, and for some $0 < r \neq i+1 < n$ we have $j_r = 0$. In particular, this tells us that there exist a $0 < r \neq i+1 < n$ such that $\mu_r = 1$ and $j_r = 0$:

$$(4.1.17) \quad \begin{array}{ccccc} & 1 & 0 & 0 & \\ \dots & \uparrow & \downarrow & \uparrow & \dots \end{array}$$

This has to hold because otherwise \hat{J} cannot satisfy condition (4.1.1). Let us assume r to be the smallest integer such that $i+1 < r$, $\mu_r = 1$ and $j_r = 0$. By our assumption that we cannot apply an arc or Y-rule to J and μ , we see that $\mu_{r-1} = 2$ and $j_{r-1} = \pm 1$. Applying an H-rule to (j_{r-1}, j_r) and (μ_{r-1}, μ_r)

$$(4.1.18) \quad \begin{array}{ccccc} & 1 & 0 & \pm 1 & 0 \\ \dots & \uparrow & \downarrow & \downarrow & \uparrow & \dots \end{array}$$

results in new sign and state strings, also with length n , satisfying condition (4.1.1). The application of the H-rule in the above case moves the zero at the r -th position to the $r-1$ -th position. Either we can now apply an arc or a Y-rule to the new sign and state strings, or by repeatedly apply

an H-rule in the manner of (4.1.10), we obtain a pair:

$$(4.1.19) \quad \begin{array}{c} 0 \quad 0 \\ \vdots \downarrow \quad \uparrow \vdots \end{array}$$

to which we can apply an arc-rule. Finally, apply induction.

It remains to show that the web w (with flow) produced from the above algorithm is an element of B_s , that is, w does not contain digons or squares. We note that, just as in [29], in the expression of w using the arc, Y and H-rules, digons can only appear as the result of applying an arc-rule to the bottom of an H-rule:

$$(4.1.20) \quad \begin{array}{c} | \\ \hline | \\ \cup \end{array}$$

A square can only result from the following sequence of arc, Y and H-rules:

$$(4.1.21) \quad \begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \quad \text{Diagram 3} \quad \text{Diagram 4} \end{array}$$

Note that in the above, we do not consider the case in which we apply an H-rule to the bottom of another H-rule. This is because such a case cannot arise in our construction of w .

Recall that in our inductive construction of w , we only apply H-rules equipped with the following flows:

$$(4.1.22) \quad \begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \end{array}$$

We can immediately see that it is not possible to apply an arc-rule with flow to the bottom of such an H-rule:

$$(4.1.23) \quad \begin{array}{c} 0 \quad \pm 1 \\ \downarrow \quad \uparrow \\ \text{Diagram} \end{array}$$

Since we only use the above two H-rules with flow, the induced flows on squares are as follows:

$$(4.1.24) \quad \begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \end{array}$$

(4.1.25)

(4.1.26)

In each case, one can check that it is possible to apply an arc-rule to the state and sign strings (the same analysis applies to the cases where the faces above are given the opposite edge orientations). However, recall that an H-rule is used in our construction only in the case for which it is not possible to apply any other rules to the boundary. This implies that none of the above faces can appear during the construction of w . \square

Implicit in the proof of Lemma 4.1.4 is a procedure to construct, from a state string J satisfying condition (4.1.1), a non-elliptic web w with flow extending J , such that $\partial w = \mu$. Note that this procedure is not deterministic. That is, it is possible to produce different webs with flows extending J by making different choices in the construction.

Example 4.1.5. *The procedure is exemplified below. If we choose to replace the substrings as indicated in the right figure, the tableau on the left gives rise to the web with flow next to it.*

(4.1.27)

1	0	-1
2	1	2
4	3	4
6	5	7

However, for other choices the same tableau generates the following web with flow:

(4.1.28)

As a matter of fact, we could also invert the orientation of the flow in the internal cycle. The resulting web with flow would still correspond to the same tableau.

However, when we restrict to semi-standard tableaux, the procedure gives a unique web with flow, the canonical flow. One can check that the procedure implicit in Lemma 4.1.4, restricts to the same bijection between Std_μ^λ and non-elliptic webs as defined by Russell in [52].

4.2. $Z(G_S)$ **and** $E(Z(G_S))$. In this subsection, S continues to be a fixed sign string of length n . Moreover, we continue to use some of the other notations and conventions from the previous subsection as well, e.g. $d = 3k \geq n$ etc. Let μ be the composition associated to S and let S_μ be the corresponding parabolic subgroup of the symmetric group S_d .

Let $Z(K_S)$ be the center of K_S and let X_μ^λ be the Spaltenstein variety, with the notation as in [8]. If $n_s = 0$, then $X_\mu^\lambda = X^\lambda$, the latter being the Springer fiber associated to λ .²

In Theorem 4.2.3, we are going to prove that $H^*(X_\mu^\lambda)$ and $Z(K_S)$ are isomorphic as graded algebras.

Recall the following result by Tanisaki [56]. Let $P = \mathbb{C}[x_1, \dots, x_d]$ and let I^λ be the ideal generated by

$$(4.2.1) \quad \left\{ e_r(i_1, \dots, i_m) \mid \begin{array}{l} m \geq 1, 1 \leq i_1 < \dots < i_m \leq d \\ r > m - \lambda_{d-m+1} - \dots - \lambda_n \end{array} \right\},$$

where $e_r(i_1, \dots, i_m) \in P$ is the r -th elementary symmetric polynomial. Write

$$R^\lambda := P/I^\lambda.$$

Tanisaki showed that

$$H^*(X^\lambda) \cong R^\lambda.$$

Note that S_μ acts on P by permuting the variables and that it maps I^λ to itself. Let $P^\mu := P^{S_\mu} \subset P$ be the subring of polynomials which are invariant under S_μ . For $1 \leq i_1 \leq \dots \leq i_m \leq n$ and $r \geq 1$, we let $e_r(\mu, i_1, \dots, i_m)$ denote the r -th elementary symmetric polynomials in the variables $X_{i_1} \cup \dots \cup X_{i_m}$, where

$$X_p := \{x_k \mid \mu_1 + \dots + \mu_{p-1} + 1 \leq k \leq \mu_1 + \dots + \mu_p\}.$$

So, we have

$$e_r(\mu, i_1, \dots, i_m) := \sum_{r_1 + \dots + r_m = r} e_{r_1}(\mu; i_1) \cdots e_{r_m}(\mu; i_m).$$

If $r = 0$, we set $e_r(\mu, i_1, \dots, i_m) = 1$ and if $r < 0$, we set $e_r(\mu, i_1, \dots, i_m) = 0$. Let I_μ^λ be the ideal generated by

$$(4.2.2) \quad \left\{ e_r(\mu, i_1, \dots, i_m) \mid \begin{array}{l} m \geq 1, 1 \leq i_1 < \dots < i_m \leq d \\ r > m - \mu_{i_1} + \dots + \mu_{i_m} - \lambda_{l+1} - \dots - \lambda_n \\ \text{where } l := \#\{i \mid \mu_i > 0, i \neq i_1, \dots, i_m\} \end{array} \right\}.$$

Note that $I_\mu^\lambda \subseteq I^\lambda$ holds. Write

$$R_\mu^\lambda := P^\mu / I_\mu^\lambda.$$

Brundan and Ostrick [8] proved that

$$H^*(X_\mu^\lambda) \cong R_\mu^\lambda.$$

First we want to show that R_μ^λ acts on K_S . Clearly, P^μ acts on K_S , by converting polynomials into dots on the facets meeting \tilde{S} .

Lemma 4.2.1. *The ideal I_μ^λ annihilates any foam in K_S .*

²When comparing to Khovanov's result for \mathfrak{sl}_2 , the reader should be aware that he labels the Springer fiber by λ^T , the transpose of λ .

Proof. The following argument demonstrates that it suffices to show this for the case when $n_s = 0$. Let $u, v \in B_S$. For each $1 \leq i \leq n$ with $s_i = -$, glue a Y onto the i -th boundary edge of u and v , respectively. Call these new webs \hat{u} and \hat{v} , respectively. Note that $\partial\hat{u} = \partial\hat{v} = \hat{S}$, where $\hat{S} = (+^d)$. Let $f \in {}_uK_v$ be any foam. For each $1 \leq i \leq n$ with $s_i = -$, glue a digon foam on top of the i -th facet of f meeting S . The new foam \hat{f} , obtained in this way, belongs to ${}_{\hat{u}}K_{\hat{v}}$. Note that we can reobtain f by capping off \hat{f} with dotted digon foams. Any polynomial $p \in I_\mu^\lambda \subseteq I^\lambda$ acting on f also acts on \hat{f} . So, if we know that $p\hat{f} = 0$, then it follows that $pf = 0$.

Thus, without loss of generality, assume that $n_s = 0$. We are now going to show that I^λ annihilates K_S .

As follows from Definition in (4.2.1), I^λ is generated by the elementary symmetric polynomials $e_r(x_{i_1}, \dots, x_{i_m})$, for the following values of m and r :

$$\begin{array}{ll} m = 2n + 1 & ; \quad r > 2n - 2, \\ m = 2n + 2 & ; \quad r > 2n - 4, \\ \vdots & \quad \quad \quad \vdots \\ m = 3n - 1 & ; \quad r > 2, \\ m = 3n & ; \quad r > 0. \end{array}$$

Note that for $m = 3n$, we simply get all completely symmetric polynomials of positive degree in the variables x_1, \dots, x_d . Any such polynomial p annihilates any foam $f \in {}_uK_v$, because by the complete symmetry of p , the dots can all be moved to the three facets around one singular edge. The relations (Dot Migration) then show that p kills f .

Now suppose $m = 3n - \ell$, for $\ell > 0$. So we must have $r > 2\ell$. The argument we are going to give does not depend on the particular choice of $i_1, \dots, i_m \subseteq \{1, 2, \dots, d\}$, so, without loss of generality, let us assume that $(i_1, \dots, i_m) = (1, \dots, m)$.

Let f be any foam in ${}_uK_v$.

First assume that $\ell = 1$.

$$\begin{aligned} & e_r(x_1, \dots, x_{d-1})f \\ &= -e_{r-1}(x_1, \dots, x_{d-1})x_d f \\ &= e_{r-2}(x_1, \dots, x_{d-1})x_d^2 f \\ & \quad \vdots \\ &= (-1)^r x_d^r f, \end{aligned}$$

All these equations follow from the fact that, for any $j > 0$, we have

$$e_j(x_1, \dots, x_d) = e_j(x_1, \dots, x_{d-1}) + e_{j-1}(x_1, \dots, x_{d-1})x_d,$$

and the fact that $e_j(x_1, \dots, x_d)f = 0$, as we proved above in the previous case for $m = 3n$. Since in this case we have $r > 2$, we see that

$$(-1)^r x_d^r f = 0,$$

by Relation (3D). This finishes the proof for this case.

In general, for $\ell \geq 1$, we get that $e_r(x_1, \dots, x_{d-\ell})f$ is equal to a linear combination of terms of the form

$$x_{d-\ell+1}^{r_1} x_{d-\ell+2}^{r_2} \cdots x_d^{r_\ell} f,$$

with $r_1 + \cdots + r_\ell = r$. Since $r > 2\ell$, there exists a $1 \leq j \leq \ell$ such that $r_j > 2$, in each term. So each term kills f , by Relation (3D). This finishes the proof. \square

Note that Lemma 4.2.1 shows that there is a well-defined homomorphism of graded algebras $c_S: R_\mu^\lambda \rightarrow Z(K_S)$, defined by

$$c_S(p) := p1.$$

Similarly, there is a filtration preserving homomorphism

$$P^\mu \rightarrow Z(G_S)$$

defined by $p \mapsto p1$. This homomorphism does not descend to R_μ^λ , because the relations in G_S are deformations of those in K_S , but the associated graded homomorphism maps P^μ to $E(Z(G_S))$ and we have

$$E(P^\mu 1) = R_\mu^\lambda 1.$$

Before giving our following result, we recall that Brundan and Ostrik [8] showed that

$$\dim H^*(X_\mu^\lambda) = \#\text{Col}_\mu^\lambda.$$

They actually gave a concrete basis, but we do not need it here.

Lemma 4.2.2. *We have*

$$\dim Z(G_S) = \#\text{Col}_\mu^\lambda.$$

Proof. Let J be any state-string satisfying condition (4.1.1). We define

$$(4.2.3) \quad z_J = \sum_{u \in B_S} \sum_T e_{u,T} \in G_S,$$

where the second sum is over all 3-colorings of u extending J .

First we show that $z_J \in Z(G_S)$. For any $u, v \in B_S$, let $f \in {}_u G_v$. Choose two arbitrary compatible colorings T_1 and T_2 of u and v , respectively. Assume that $e_{T_1} f e_{T_2} \neq 0$. Then we have

$$z_J e_{T_1} f e_{T_2} = \begin{cases} e_{T_1} f e_{T_2}, & \text{if } T_1 \text{ extends } J, \\ 0 & \text{else.} \end{cases}$$

We also have

$$e_{T_1} f e_{T_2} z_J = \begin{cases} e_{T_1} f e_{T_2}, & \text{if } T_2 \text{ extends } J, \\ 0 & \text{else.} \end{cases}$$

This shows that $z_J \in Z(G_S)$, because T_1 and T_2 are compatible, and so T_1 extends J if and only if T_2 extends J .

Note that

$$\sum_J z_J = 1 \quad \text{and} \quad z_J z_{J'} = \delta_{J,J'} z_J.$$

In particular, the z_J 's are linearly independent.

For any state-string J satisfying condition (4.1.1), the central idempotent z_J belongs to $P^\mu 1$. In order to see this, first note that, for any $u \in B_S$, the element

$$z_J 1_u = \sum_{\substack{T \text{ extends } J \\ 41}} e_{u,T},$$

belongs to $P^\mu 1_u$. This holds, because only the colors of the boundary edges of u are fixed. We can sum over all possible 3-colorings of the other edges, which implies that these edges only contribute a factor 1 to $z_J 1_u$. Furthermore, we see that $z_J 1_u = p_J 1_u$, for a fixed polynomial $p_J \in P^\mu$, i.e. p_J is independent of u . Therefore, we have

$$z_J = \sum_{u \in B_S} p_J 1_u = p_J 1 \in P^\mu 1.$$

It remains to show that $Z(G_S) z_J = \mathbb{C} z_J$. Let $z \in Z(G_S)$. By the orthogonality of Gornik's symmetric idempotents, we have

$$z = \sum_{u,T} e_{u,T} z e_{u,T}.$$

By Theorem 3.0.25, we know that

$$e_{u,T} z e_{u,T} = \lambda_{u,T}(z) e_{u,T},$$

for a certain $\lambda_{u,T}(z) \in \mathbb{C}$. Therefore, we have

$$z = \sum_{u,T} \lambda_{u,T}(z) e_{u,T} \in \bigoplus_{u,T} \mathbb{C} e_{u,T}.$$

By Lemma 4.1.4, we know that $z_J \neq 0$. This shows that

$$\{z_J \mid J \text{ satisfying condition (4.1.1)}\}$$

forms a basis of $Z(G_S)$. By Proposition 4.1.2, the claim of the lemma follows. \square

Theorem 4.2.3. *The degree preserving algebra homomorphism*

$$c_S: R_{\mu_S}^\lambda \rightarrow Z(K_S)$$

is an isomorphism.

Proof. In Corollary 5.3.11 it will be shown that

$$\dim H^*(X_{\mu_S}^\lambda) = \dim Z(K_S),$$

so it suffices to show that c_S is injective.

Lemma 4.2.1 shows that (as graded complex algebras)

$$R_\mu^\lambda 1 \subset Z(K_S).$$

As already mentioned above, Brundan and Ostrik [8] showed that

$$H^*(X_\mu^\lambda) \cong R_\mu^\lambda$$

as graded complex algebras.

The proof of Lemma 4.2.2 shows that the filtration preserving homomorphism

$$P^\mu \rightarrow Z(G_S),$$

defined by $p \mapsto p1$, is surjective. Note the $E(-)$ is not a map. However, a filtered algebra A and its associated graded $E(A)$ are isomorphic as vector spaces. In particular, they satisfy

$$\dim A = \dim E(A).$$

Therefore, since $p \mapsto p1$ is a surjection of vector spaces, we have

$$\dim Z(G_S) = \dim E(Z(G_S)) = \dim E(P^\mu 1) = \dim P^\mu 1.$$

Recall that $E(P^\mu 1) = R_\mu^\lambda 1$ and $\dim Z(G_S) = \dim R_\mu^\lambda$. This shows

$$\dim R_\mu^\lambda 1 = \dim P^\mu 1 = \dim Z(G_S) = \dim R_\mu^\lambda,$$

which implies that the map c_S is injective. \square

5. WEB ALGEBRAS AND THE CYCLOTOMIC KLR ALGEBRAS

5.1. Howe duality. Our main references for Howe duality are [22] and [23], where the reader can find the proofs of the results, which we recall below, and other details.

Let us briefly explain Howe duality.³ The natural actions of $\mathrm{GL}_m := \mathrm{GL}(m, \mathbb{C})$ and $\mathrm{GL}_n := \mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ commute and the two groups are each others commutant. We say that the actions of GL_m and GL_n are *Howe dual*.

More interestingly, their actions on the symmetric powers

$$S^p(\mathbb{C}^m \otimes \mathbb{C}^n)$$

and on the alternating powers

$$\Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n)$$

are also Howe dual, for any $p \in \mathbb{N}$. These are called the *symmetric* and the *skew* Howe duality of GL_m and GL_n , respectively. In this paper, we are only interested in the skew Howe duality.

The skew Howe duality implies that we have the following decomposition into irreducible $\mathrm{GL}_m \times \mathrm{GL}_n$ -modules:

$$(5.1.1) \quad \Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\lambda} V_{\lambda} \otimes W_{\lambda'},$$

where λ ranges over all partitions with p boxes and at most m rows and n columns and λ' is the transpose of λ . Here V_{λ} is the unique irreducible GL_m -module of highest weight λ and $W_{\lambda'}$ is the unique irreducible GL_n -module of highest weight λ' . Without giving a full proof of (5.1.1), which can be found in Section 4.1 of [23], we note that it is easy to write down the highest weight vectors in the decomposition of

$$\Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n).$$

Define

$$\epsilon_{ij} := \epsilon_i \otimes \epsilon_j,$$

for any $1 \leq i \leq m$ and $1 \leq j \leq n$. Here the ϵ_i and the ϵ_j are the canonical basis elements of \mathbb{C}^m and \mathbb{C}^n respectively. Let λ be one of the highest GL_m weights in (5.1.1). Write $\lambda = (\lambda_1, \dots, \lambda_m)$ with $n \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$. Then

$$\begin{aligned} v_{\lambda, \lambda'} &:= (\epsilon_{11} \wedge \dots \wedge \epsilon_{1\lambda_1}) \wedge (\epsilon_{21} \wedge \dots \wedge \epsilon_{2\lambda_2}) \wedge (\epsilon_{m1} \wedge \dots \wedge \epsilon_{m\lambda_m}) \\ &= \pm (\epsilon_{11} \wedge \dots \wedge \epsilon_{\lambda'_1 1}) \wedge (\epsilon_{12} \wedge \dots \wedge \epsilon_{\lambda'_2 2}) \wedge (\epsilon_{1n} \wedge \dots \wedge \epsilon_{\lambda'_n n}) \end{aligned}$$

is a highest $\mathrm{GL}_m \times \mathrm{GL}_n$ weight. By convention, we exclude factors ϵ_{ij} for which $\lambda_i = 0$ or $\lambda'_j = 0$.

Now restrict to SL_m and assume that $p = mk$, for some $k \in \mathbb{N}$. By Schur's lemma, the decomposition in (5.1.1) implies that

$$(5.1.2) \quad \mathrm{Inv}_{\mathrm{GL}_m}(\Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n)) \cong \mathrm{Hom}_{\mathrm{SL}_m}(\mathbb{C}, \Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n)) \cong W_{(k^m)},$$

where \mathbb{C} denotes the trivial representation.

³We follow Kamnitzer's exposition in "The ubiquity of Howe duality", which is available at <https://sbseminar.wordpress.com/2007/08/10/the-ubiquity-of-howe-duality/>.

Decompose

$$\mathbb{C}^n \cong \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2 \oplus \cdots \oplus \mathbb{C}\epsilon_n$$

into its one-dimensional \mathfrak{gl}_n -weight spaces. Then we have

$$(5.1.3) \quad \Lambda^p(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{(p_1, \dots, p_n) \in \Lambda(n, p)} \Lambda^{p_1}(\mathbb{C}^m) \otimes \Lambda^{p_2}(\mathbb{C}^m) \otimes \cdots \otimes \Lambda^{p_n}(\mathbb{C}^m)$$

as $\mathrm{GL}_m \times T$ -modules, where T is the diagonal torus in GL_n .

This decomposition implies that

$$(5.1.4) \quad \mathrm{Inv}_{\mathrm{SL}_m}(\Lambda^{p_1}(\mathbb{C}^m) \otimes \Lambda^{p_2}(\mathbb{C}^m) \otimes \cdots \otimes \Lambda^{p_n}(\mathbb{C}^m)) \cong W(p_1, \dots, p_n),$$

where $W(p_1, \dots, p_n)$ denotes the (p_1, \dots, p_n) -weight space of $W_{(k^m)}$.

Cautis has written down a q -version of skew Howe duality in Section 6.1 in [10] (see also [12]). We do not recall his general explanation here. Instead, in the next subsection, we use Kuperberg's webs to give a q -version of the isomorphism in (5.1.4), for $U_q(\mathfrak{sl}_3)$ and $U_q(\mathfrak{gl}_n)$ with $n = 3k$ and $k \in \mathbb{N}$ arbitrary but fixed.

We also categorify this instance of q -skew Howe duality, as we will explain after the next subsection.

5.2. The uncategorified story.

5.2.1. Enhanced sign sequences. In this section we slightly generalize the notion of a sign sequence/string. We call this generalization *enhanced sign sequence* or *enhanced sign string*. Note that, with a slight abuse of notation, we use \hat{S} for sign strings and S for enhanced sign string throughout the whole section.

Definition 5.2.1. An *enhanced sign sequence/string* is a sequence $S = (s_1, \dots, s_n)$ with entries $s_i \in \{\circ, -1, +1, \times\}$, for all $i = 1, \dots, n$. The corresponding weight $\mu = \mu_S \in \Lambda(n, d)$ is given by the rules

$$\mu_i = \begin{cases} 0, & \text{if } s_i = \circ, \\ 1, & \text{if } s_i = 1, \\ 2, & \text{if } s_i = -1, \\ 3, & \text{if } s_i = \times. \end{cases}$$

Let $\Lambda(n, d)_3 \subset \Lambda(n, d)$ be the subset of weights with entries between 0 and 3. Recall that $\Lambda(n, d)_{1,2}$ denotes the subset of weights with only 1 and 2 as entries.

Let $n = d = 3k$. For any enhanced sign string S such that $\mu_S \in \Lambda(n, n)_3$, we define \hat{S} to be the sign sequence obtained from S by deleting all entries that are equal to \circ or \times and keeping the linear ordering of the remaining entries. Similarly, for any $\mu \in \Lambda(n, n)_3$, let $\hat{\mu}$ be the weight obtained from μ by deleting all entries which are equal to \circ or 3 . Thus, if $\mu = \mu_S$, for a certain enhanced sign string S , then $\hat{\mu} = \mu_{\hat{S}}$. Note that $\hat{\mu} \in \Lambda(m, d)_{1,2}$, for a certain $0 \leq m \leq n$ and $d = 3(k - (n - m))$.

Note that for any semi-standard tableau $T \in \mathrm{Std}_{\mu}^{(3^k)}$, there is a unique semi-standard tableau $\hat{T} \in \mathrm{Std}_{\hat{\mu}}^{(3^{k-(n-m)})}$, obtained by deleting any cell in T whose label appears three times and keeping the linear ordering of the remaining cells within each column.

Conversely, let $\mu' \in \Lambda(m, d)_{1,2}$, with $m \leq n$ and $d = 3(k - (n - m))$. In general, there is more than one $\mu \in \Lambda(n, n)_3$ such that $\hat{\mu} = \mu'$, but at least one. Choose one of them, say μ_0 . Then, given any $T' \in \text{Std}_{\mu'}^{(3^k - (n - m))}$, there is a unique $T \in \text{Std}_{\mu_0}^{(3^k)}$ such that $\hat{T} = T'$.

The construction of T is as follows: suppose that i is the smallest number such that $(\mu_0)_i = 3$.

- (1) In each column c of T' , there is a unique vertical position such that all cells above that position have label smaller than i and all cells below that position have label greater than i . Insert a new cell labeled i precisely in that position, for each column c .
- (2) In this way, we obtain a new tableau of shape $(3^k - (n - m) + 1)$. It is easy to see that this new tableau is semi-standard. Now apply this procedure recursively for each $i = 1, \dots, n$, such that $(\mu_0)_i = 3$.
- (3) In this way, we obtain a tableau T of shape (3^k) . Since in each step the new tableau that we get is semi-standard, we see that T belongs to $\text{Std}_{\mu_0}^{(3^k)}$.

Note also that $\hat{T} = T'$. This shows that for a fixed $\mu \in \Lambda(n, n)_3$, we have a bijection

$$\text{Std}_{\mu}^{(3^k)} \ni T \longleftrightarrow \hat{T} \in \text{Std}_{\hat{\mu}}^{(3^k - (n - m))}.$$

Given an enhanced sign sequence S , such that $\mu_S \in \Lambda(n, n)_3$, we define

$$W_S := W_{\hat{S}}.$$

In other words, as a vector space W_S does not depend on the \circ and \times -entries of S . However, they do play an important role below. Similarly, we define

$$B_S := B_{\hat{S}} \quad \text{and} \quad K_S := K_{\hat{S}}.$$

5.2.2. An instance of q -skew Howe duality. Let $V_{(3^k)}$ be the irreducible $U_q(\mathfrak{gl}_n)$ -module of highest weight (3^k) . By restriction, $V_{(3^k)}$ is also a $U_q(\mathfrak{sl}_n)$ -module and, since it is a weight representation, it is a $\dot{U}(\mathfrak{sl}_n)$ -module, too. It is well-known (see [20] and [44]) for example) that

$$\dim V_{(3^k)} = \sum_{\mu \in \Lambda(n, n)_3} \#\text{Std}_{\mu}^{(3^k)}.$$

Note that a tableau of shape (3^k) can only be semi-standard if its filling belongs to $\Lambda(n, n)_3$, so strictly speaking we could drop the 3-subscript. More precisely, if

$$V_{(3^k)} = \bigoplus_{\mu \in \Lambda(n, n)_3} V_{(3^k)}(\mu)$$

is the $U_q(\mathfrak{gl}_n)$ weight decomposition of $V_{(3^k)}$, then

$$\dim V_{(3^k)}(\mu) = \#\text{Std}_{\mu}^{(3^k)}.$$

Note that the action of $U_q(\mathfrak{gl}_n)$ on $V_{(3^k)}$ descends to $S_q(n, n)$ and recall that there exists a surjective algebra homomorphism

$$\psi_{n,n}: \dot{U}(\mathfrak{sl}_n) \rightarrow S_q(n, n).$$

The action of $\dot{U}(\mathfrak{sl}_n)$ on $V_{(3^k)}$ is equal to the pull-back of the action of $S_q(n, n)$ via $\psi_{n,n}$.

Define

$$W_{(3^k)} := \bigoplus_{S \in \Lambda(n, n)_3} W_S.$$

Below, we will show that $S_q(n, n)$ acts on $W_{(3^k)}$. Pulling back the action via $\psi_{n,n}$, we see that $W_{(3^k)}$ is a $\dot{U}(\mathfrak{sl}_n)$ -module. We will also show that

$$W_{(3^k)} \cong V_{(3^k)}$$

as $S_q(n, n)$ -modules, and therefore also as $\dot{U}(\mathfrak{sl}_n)$ -modules, and that W_S corresponds to the μ_S -weight space of $V_{(3^k)}$.

Let us define the aforementioned left action of $S_q(n, n)$ on $W_{(3^k)}$. The reader should compare this action to the categorical action on the objects in Section 4.2 in [41]. Note that our conventions in this paper are different from those in [41].

Definition 5.2.2. Let

$$\phi: S_q(n, n) \rightarrow \text{End}_{\mathbb{C}(q)}(W_{(3^k)})$$

be the homomorphism of $\mathbb{C}(q)$ -algebras defined by gluing the following webs on top of the elements in $W_{(3^k)}$:

$$\begin{array}{c} 1_\lambda \mapsto \begin{array}{c} \begin{array}{c} | \quad | \quad \dots \quad | \\ \lambda_1 \quad \lambda_2 \quad \quad \lambda_n \end{array} \end{array} \\ \\ E_{\pm i} 1_\lambda \mapsto \begin{array}{c} \begin{array}{c} \lambda_i \pm 1 \quad \lambda_{i+1} \mp 1 \\ | \quad \dots \quad | \quad \text{---} \quad | \quad \dots \quad | \\ \lambda_1 \quad \lambda_{i-1} \quad \lambda_i \quad \lambda_{i+1} \quad \lambda_{i+2} \quad \lambda_n \end{array} \end{array} \end{array}$$

We use the convention that vertical edges labeled 1 are oriented upwards, vertical edges labeled 2 are oriented downwards and edges labeled 0 or 3 are erased. The orientation of the horizontal edges is uniquely determined by the orientation of the vertical edges. With these conventions, one can check that the horizontal edge is always oriented from right to left for E_{+i} and from left to right for E_{-i} .

Furthermore, let $\lambda \in \Lambda(n, n)$ and let S be any sign string such that $\mu_S \in \Lambda(n, n)_3$. For any $w \in W_S$, we define

$$\phi(1_\lambda)w = 0, \quad \text{if } \mu_S \neq \lambda.$$

By $\phi(1_\lambda)w$ we mean the left action of $\phi(1_\lambda)$ on w . In particular, for any $\lambda > (3^k)$, we have $\phi(1_\lambda) = 0$ in $\text{End}_{\mathbb{C}(q)}(W_{(3^k)})$.

Let us give two examples to show how these conventions work. We only write down the relevant entries of the weights and only draw the important edges. We have

$$\begin{array}{c}
 E_{+1}1_{(22)} \mapsto \begin{array}{c} \begin{array}{c} 3 \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \uparrow \\ 2 \end{array} \\ \begin{array}{c} \downarrow \\ 2 \end{array} \begin{array}{c} 0 \\ \downarrow \\ 2 \end{array} \end{array} \\
 E_{-2}E_{+1}1_{(121)} \mapsto \begin{array}{c} \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array} \begin{array}{c} 2 \\ \uparrow \\ 2 \end{array} \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array} \end{array}
 \end{array}$$

Remark 5.2.3. Note that the introduction of enhanced sign strings is necessary for the definition of ϕ to make sense. Although as a vector space W_S does not depend on the entries of S which are equal to \circ or \times , the $S_q(n, n)$ action on W_S does depend on them.

Remark 5.2.4. A more general version of the map ϕ is studied in the forthcoming paper [12] by Cautis, Kamnitzer, and Morrison.

Lemma 5.2.5. *The map ϕ in Definition 5.2.2 is well-defined.*

Proof. It follows immediately from its definition that ϕ preserves the three relations (2.3.3), (2.3.4) and (2.3.5).

Checking case by case, one can easily show that ϕ preserves (2.3.6) by using the relations (2.1.3), (2.1.4) and (2.1.5). We do just one example and leave the other cases to the reader. The figure below shows the image of the relation

$$E_1 E_{-1} 1_{(21)} - E_{-1} E_1 1_{(21)} = 1_{(21)}$$

under ϕ .

$$\begin{array}{c} \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \uparrow \\ 1 \end{array} \end{array} - \begin{array}{c} \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \uparrow \\ 1 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 2 \\ \downarrow \\ 2 \end{array} \begin{array}{c} 1 \\ \uparrow \\ 1 \end{array} \end{array}$$

This relation is exactly the third Kuperberg relation in (2.1.5). □

Lemma 5.2.6. *The map ϕ gives rise to an isomorphism*

$$\phi: V_{(3^k)} \rightarrow W_{(3^k)}$$

of $S_q(n, n)$ -modules.

Proof. Note that the empty web $w_h := w_{(3^k)}$, which generates $W_{(\times^k, 0^{2k})} \cong \mathbb{C}(q)$, is a highest weight vector.

The map ϕ induces a surjective homomorphism of $S_q(n, n)$ -modules

$$\phi: S_q(n, n)1_{(3^k)} \rightarrow W_{(3^k)},$$

defined by

$$\phi(x1_{(3^k)}) := \phi(x)w_h.$$

As we already remarked above, we have

$$\begin{aligned}\dim V_{(3^k)} &= \sum_{\mu_S \in \Lambda(n, n)_3} \# \text{Std}_{\mu_S}^{(3^k)} \\ &= \sum_{\mu_S \in \Lambda(n, n)_3} \dim W_S = \dim W_{(3^k)}.\end{aligned}$$

Therefore, we have

$$V_{(3^k)} \cong \phi(S_q(n, n)) w_h \cong W_{(3^k)},$$

which finishes the proof.

It is well-known that

$$V_{(3^k)} \cong S_q(n, n) 1_{(3^k)} / (\mu > (3^k)),$$

where $(\mu > (3^k))$ is the ideal generated by all elements of the form $x 1_{\mu} y 1_{(3^k)}$, such that $x, y \in S_q(n, n)$ and μ is some weight greater than (3^k) . This quotient of $S_q(n, n)$ is an example of a so called *Weyl module*. We see that the kernel of ϕ is also equal to $(\mu > (3^k))$. \square

We want to explain two more facts about the isomorphism in Lemma 5.2.6, which we will need later in this paper.

Recall that there is an inner product on $V_{(3^k)}$. First of all, there is an involution on $\mathbb{C}(q)$ determined by

$$\overline{aq^n} = \bar{a} q^{-n},$$

for any $a \in \mathbb{C}$. Here \bar{a} denotes the complex conjugate of a . Recall Lusztig's $\mathbb{C}(q)$ antilinear (antilinear means w.r.t. to the involution above) algebra anti-involution τ on $S_q(n, n)$ defined by

$$\tau(1_{\lambda}) = 1_{\lambda}, \quad \tau(1_{\lambda+\alpha_i} E_i 1_{\lambda}) = q^{-1-\bar{\lambda}_i} 1_{\lambda} E_{-i} 1_{\lambda+\alpha_i}, \quad \tau(1_{\lambda} E_{-i} 1_{\lambda+\alpha_i}) = q^{1+\bar{\lambda}_i} 1_{\lambda+\alpha_i} E_i 1_{\lambda}.$$

The q -Shapovalov form $\langle \cdot, \cdot \rangle$ on $V_{(3^k)}$ is the unique $\mathbb{C}(q)$ sesquilinear form such that

- (1) $\langle v_h, v_h \rangle = 1$, for a fixed highest weight vector v_h .
- (2) $\langle xv, v' \rangle = \langle v, \tau(x)v' \rangle$, for any $x \in S_q(n, n)$ and any $v, v' \in V_{(3^k)}$.
- (3) $f \langle v, v' \rangle = \langle v \bar{f}, v' \rangle = \langle v, v' f \rangle$, for any $f \in \mathbb{C}(q)$ and any $v, v' \in V_{(3^k)}$.

We can also define an inner product on $W_{(3^k)}$, using the Kuperberg bracket. Let S be any enhanced sign string S , such that $\mu_S \in \Lambda(n, n)_3$. Denote the length of the sign string \hat{S} by $\ell(\hat{S})$.

Definition 5.2.7. Define the $\mathbb{C}(q)$ sesquilinear *normalized Kuperberg form* by

- $\langle w_h, w_h \rangle = 1$;
- $\langle u, v \rangle := q^{\ell(\hat{S})} \langle u^* v \rangle_{\text{Kup}}$;
- $\langle f(q)u, g(q)v \rangle := \overline{f(q)} g(q) \langle u, v \rangle$,

for any $u, v \in B_S$ and $f(q), g(q) \in \mathbb{C}(q)$.

The following lemma motivates the normalization of the Kuperberg form.

Lemma 5.2.8. *The isomorphism of $S_q(n, n)$ -modules*

$$\phi: V_{(3^k)} \rightarrow W_{(3^k)}$$

is an isometry.

Proof. First note that

$$\langle (E_{\pm i} u)^* v \rangle_{\text{Kup}} = \langle u^* E_{\mp i} v \rangle_{\text{Kup}},$$

for any $u, v \in W_S$ and any $i = 1, \dots, n$, which is exactly (2) from above. This shows that the result of the lemma holds up to normalization.

Our normalization of the Kuperberg form matches the normalization of the q -Shapovalov form. One can easily check this case by case. Let us just do two examples. Let $i = 1$. Then one has $E_1 1_{(a,b,\dots)} = 1_{(a+1,b-1,\dots)} E_1$. If $(a, b, \dots) \in \Lambda(n, n)_3$ such that $a - b = -1$, then

$$\ell(\widehat{(a, b)}) = \ell(\widehat{(a+1, b-1)}),$$

where ℓ indicates the length of the sign sequence. This matches

$$\tau(E_1 1_{(a,b)}) = 1_{(a,b)} E_{-1}.$$

If $(a, b) = (2, 1)$, then $E_1 1_{(2,1,\dots)} = 1_{(3,0,\dots)} E_1$. Note that

$$\ell(\widehat{(2, 1, \dots)}) = \ell(\widehat{(3, 0, \dots)}) + 2.$$

This $+2$ cancels exactly with the -2 , which appears as the exponent of q in

$$\tau(E_1 1_{(2,1,\dots)}) = q^{-2} 1_{(2,1,\dots)} E_{-1}.$$

□

We will need one more fact about ϕ . For any $i = 1, \dots, n$ and any $a \in \mathbb{N}$, let

$$E_{\pm i}^{(a)} := \frac{E_{\pm i}^a}{[a]!}$$

denote the *divided power* in $S_q(n, n)$. Recall the following relations for the divided powers:

$$(5.2.1) \quad E_{\pm i}^{(a)} E_{\pm i}^{(b)} 1_\lambda = \begin{bmatrix} a+b \\ a \end{bmatrix} E_{\pm i}^{(a+b)} 1_\lambda,$$

$$(5.2.2) \quad E_{+i}^{(a)} E_{-i}^{(b)} 1_\lambda = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+\lambda_i-\lambda_{i+1} \\ j \end{bmatrix} E_{-i}^{(b-j)} E_{+i}^{(a-j)} 1_\lambda,$$

$$(5.2.3) \quad E_{-i}^{(b)} E_{+i}^{(a)} 1_\lambda = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} b-a-(\lambda_i-\lambda_{i+1}) \\ j \end{bmatrix} E_{+i}^{(a-j)} E_{-i}^{(b-j)} 1_\lambda.$$

Here $[a]!$ denotes the *quantum factorial* and $\begin{bmatrix} a \\ b \end{bmatrix}$ denotes the *quantum binomial*.

The images of the divided powers under

$$\phi: S_q(n, n) \rightarrow \text{End}(W_{(3^k)})$$

are easy to compute. For example, we have (for simplicity, we only draw two of the strands and write $E = E_{+i}$)

$$\phi(E^2 1_{(0,2)}) = \begin{array}{c} \begin{array}{cc} 2 & 0 \\ | & | \\ 1 & 1 \\ | & | \\ 0 & 2 \end{array} \end{array} = \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \diagdown & \diagup \\ & \text{loop} \\ \diagup & \diagdown \\ \circ & \circ \end{array} \end{array} = [2] \begin{array}{c} \begin{array}{cc} \circ & \circ \\ \diagdown & \diagup \\ & \text{crossing} \\ \diagup & \diagdown \\ \circ & \circ \end{array} \end{array}.$$

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Therefore, we get

$$\phi(E^{(2)}1_{(0,2)}) = \begin{array}{c} - \quad \circ \\ \searrow \quad \nearrow \\ \circ \quad - \end{array}.$$

Another interesting example is

$$\phi(E^2 1_{(0,3)}) = \begin{array}{c} 2 \quad 1 \\ | \quad | \\ 1 \text{---} 2 \\ | \quad | \\ 0 \quad 3 \end{array} = [2] \begin{array}{c} - \quad + \\ \searrow \quad \nearrow \\ \circ \quad \times \end{array},$$

which shows that

$$\phi(E^{(2)}1_{(0,3)}) = \begin{array}{c} - \quad + \\ \searrow \quad \nearrow \\ \circ \quad \times \end{array}.$$

The final example we will consider is $\phi(E^{(3)}1_{(0,3)})$. We see that

$$\phi(E^3 1_{(0,3)}) = \begin{array}{c} 3 \quad 0 \\ | \quad | \\ 2 \text{---} 1 \\ | \quad | \\ 1 \text{---} 2 \\ | \quad | \\ 0 \quad 3 \end{array} = \begin{array}{c} \times \quad \circ \quad \times \quad \circ \\ \circ \quad \times \quad \circ \quad \times \end{array} \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \end{array} = [3]!.$$

Thus, we have

$$\phi(E^{(3)}1_{(0,3)}) = \begin{array}{c} \times \quad \circ \\ \circ \quad \times \end{array},$$

which is the unique empty web from (\circ, \times) to (\times, \circ) .

Note that (5.2.2) and (5.2.3) imply that, for any $a \in \mathbb{N}$, we have

$$(5.2.4) \quad E_{-i}^{(a)} E_{+i}^{(a)} 1_{(\dots, 0, a, \dots)} = 1_{(\dots, 0, a, \dots)} \quad \text{and} \quad E_{+i}^{(a)} E_{-i}^{(a)} 1_{(\dots, a, 0, \dots)} = 1_{(\dots, a, 0, \dots)}$$

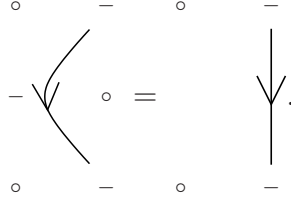
in $S_q(n, n)$. Similarly, let $S_q(n, n)/I$, where I denotes the two-sided ideal generated by all 1_μ such that $\mu > (3^k)$. Again by (5.2.2) and (5.2.3), we have

$$(5.2.5) \quad E_{-i}^{(3-a)} E_{+i}^{(3-a)} 1_{(\dots, a, 3, \dots)} = 1_{(\dots, a, 3, \dots)} \quad \text{and} \quad E_{+i}^{(3-a)} E_{-i}^{(3-a)} 1_{(\dots, 3, a, \dots)} = 1_{(\dots, 3, a, \dots)}$$

in $S_q(n, n)/I$. One can check that ϕ maps the two sides of the equations in (5.2.4) and (5.2.5) to isotopic diagrams. For example, ϕ maps

$$E_{-}^{(2)} E_{+}^{(2)} 1_{(0,2)} = 1_{(0,2)}$$

to



Remark 5.2.9. Let

$$W_{(3^k)}^{\mathbb{Z}} := \bigoplus_{\mu_S \in \Lambda(n, n)_3} W_S^{\mathbb{Z}}$$

be the integral form. Then the remarks above show that the action in Definition 5.2.2 restricts to a well-defined action of $S_q^{\mathbb{Z}}(n, n)$ on $W_{(3^k)}^{\mathbb{Z}}$. Therefore, the isomorphism in Lemma 5.2.6 restricts to a well-defined isomorphism between the integral forms

$$V_{(3^k)}^{\mathbb{Z}} \cong W_{(3^k)}^{\mathbb{Z}}.$$

The proof of the following lemma is based on an algorithm, which we call *enhanced inverse growth algorithm*. The result is needed later to show surjectivity in Theorem 5.3.8.

Lemma 5.2.10. *Let S be any enhanced sign string such that $\mu_S \in \Lambda(n, n)_3$. For any $w \in B_S$, there exists a product of divided powers x , such that*

$$\phi(x1_{(3^k)}) = w.$$

Proof. Choose any $w \in B_S$. We consider $w \in B_{(\times^k, \circ^{2k})}^S$, i.e. a non-elliptic web with (empty) lower boundary determined by (\times^k, \circ^{2k}) and upper boundary determined by S . Express w using the growth algorithm, in an arbitrary way. Suppose there are m steps in this instance of the growth algorithm. The element x is built up in $m + 2$ steps: an initial step, one step for each step in the growth algorithm, and a last step. During the construction of x , we always keep track of the \circ s and \times s. At each step the strands of w are numbered according to their position in x .

If the H , Y or arc-move is applied to two non-consecutive strands, we first have to apply some divided powers, as in (5.2.4) and (5.2.5), to make them consecutive. Let $x_k \in S_q(n, n)$ be the element assigned to the k -th step and let μ^k be the weight after the k -step, i.e. $x_k = 1_{\mu^{k-1}} x_k 1_{\mu^k}$. The element x we are looking for is the product of all x_k .

- (1) Take $x_0 = 1_{\mu_S}$.
- (2) Suppose that the k -th step in the growth algorithm is applied to the strands i and $i + r$, for some $r \in \mathbb{N}_{>0}$. This means that the entries of μ^{k-1} satisfy $\mu_j \in \{0, 3\}$, for all $j = i + 1, \dots, i + r - 1$. Let x'_k be the product of divided powers which “swap” the $(\mu_{i+1}, \dots, \mu_{i+r-1})$ and μ_{i+r} . So, we first swap μ_{i+r-1} and μ_{i+r} , then μ_{i+r-2} and μ_{i+r} etc. Now, the rule in the growth algorithm, still corresponding to the k -th step, can be applied to the strands i and $i + 1$.
- (3) Suppose that it is an H -rule. If the bottom of the H is a pair (up-arrow down-arrow), then take $x_k := x'_k E_{+i}$. If the bottom of the H is a pair (down-arrow up-arrow), then take $x_k := X'_k E_{-i}$.
- (4) Suppose that the rule, corresponding to the k -th step in the growth algorithm, is a Y -rule. If the bottom strand of Y is oriented downward, then take $x_k := x'_k E_{-i}$. If it is oriented upward, take $x_k := x'_k E_{+i}$. Note that these two choices are not unique. They depend on where you put 0 or 3 in μ^k . The choice we made corresponds to taking $(\mu_i^k, \mu_{i+1}^k) = (2, 0)$

in the first case and $(\mu_i^k, \mu_{i+1}^k) = (1, 3)$ in the second case. Other choices would be perfectly fine and would lead to equivalent elements in $S_q(n, n)1_{(3^k)}/(\mu > (3^k))$.

- (5) Suppose that the rule, corresponding to the k -th step in the growth algorithm, is an arc-rule. If the arc is oriented clockwise, take $x_k := x'_k E_{-i}^{(2)}$. If the arc is oriented counter-clockwise, take $x_k = x'_k E_{-i}$. Again, these choices are not unique. They correspond to taking $(\mu_i^k, \mu_{i+1}^k) = (3, 0)$ in both cases.
- (6) After the m -th step in the growth algorithm, which is the last one, we obtain μ^m , which is a sequence of 3s and 0s. Let x_{m+1} be the product of divided powers which reorders the entries of μ^m , so that $\mu^{m+1} = (3^k)$.
- (7) Take $x := 1_{\mu_S} x_1 x_2 \cdots x_{m+1} 1_{(3^k)} \in S_q(n, n)$. Note that x is of the form $E_{\underline{i}} 1_{(3^k)}$.

From the analysis of the images of the divided powers under ϕ , it is clear that

$$\phi(x) = w.$$

□

We do a simple example to illustrate Lemma 5.2.10. Let

$$w = \begin{array}{c} 1 \quad 1 \quad 1 \\ \nearrow \quad \nearrow \quad \nearrow \\ \searrow \quad \nearrow \quad \nearrow \end{array}$$

Then the algorithm in the proof of Lemma 5.2.10 gives

$$x = 1_{(111)} E_{-1} E_{-2} E_{-1} 1_{(300)},$$

or as a picture (read from bottom to top)

$$\begin{array}{c} 1 \quad 1 \quad 1 \\ \uparrow \quad \uparrow \quad \uparrow \\ E_{-1} \quad \cdots \quad 0 \quad \cdots \quad 1 \\ \uparrow \quad \uparrow \quad \uparrow \\ E_{-2} \quad \cdots \quad 1 \quad \cdots \quad 0 \\ \uparrow \quad \uparrow \quad \uparrow \\ E_{-1} \quad \cdots \quad 0 \quad \cdots \quad 0 \\ 3 \quad 0 \quad 0 \end{array}$$

We are now ready to start explaining the categorified story.

5.3. And its categorification. Let $K_S\text{-pmod}_{\text{gr}}$ be the category of all finite-dimensional projective unitary graded K_S -modules and $K_0(K_S) := K_0(K_S\text{-pmod}_{\text{gr}})$ its split Grothendieck group. Recall that a *unitary* module is one on which the identity of K_S acts as the identity operator. In what follows, it will sometimes be useful to consider homomorphisms of arbitrary degree, so we define

$$\text{HOM}_B(M, N) := \bigoplus_{t \in \mathbb{Z}} \text{Hom}_B(M, N\{t\}),$$

for any finite-dimensional associative unital graded algebra B and any finite-dimensional unitary graded B -modules M and N . Note that for almost all $t \in \mathbb{Z}$ we have $\text{Hom}_B(M, N\{t\}) = \{0\}$, so $\text{HOM}_B(M, N)$ is still finite-dimensional.

Moreover, we need the following notions throughout the rest of the section.

Suppose that S is an enhanced sign string such that $\mu_S \in \Lambda(n, n)_3$. For any $u \in B_S$, let

$$P_u := \bigoplus_{w \in B_S} {}_u K_w.$$

Then we have

$$K_S = \bigoplus_{u \in B_S} P_u,$$

and so P_u is an object in $K_S\text{-pmod}_{\text{gr}}$, for any $u \in B_S$. Note that, for any $u, v \in B_S$, we have

$$\text{HOM}(P_u, P_v) \cong {}_u K_v,$$

where an element in ${}_u K_v$ acts on P_u by composition on the right-hand side.

Similarly, we can define

$${}_u P := \bigoplus_{w \in B_S} {}_u K_w,$$

which is a right graded projective K_S -module.

Remark 5.3.1. Just one warning: the reader should not confuse P_u with $P_{u,T}$ in Section 3.

5.3.1. *The definition of $\mathcal{W}_{(3^k)}$.* Recall that S denotes an enhanced sign string. Define

$$K_{(3^k)} := \bigoplus_{\mu_S \in \Lambda(n, n)_3} K_S$$

and

$$\mathcal{W}_{(3^k)} := K_{(3^k)}\text{-pmod}_{\text{gr}} \cong \bigoplus_{\mu_S \in \Lambda(n, n)_3} K_S\text{-pmod}_{\text{gr}}.$$

The main goal of this section is to show that there exists a categorical $\mathcal{U}(\mathfrak{sl}_n)$ -action on $\mathcal{W}_{(3^k)}$ and that

$$\mathcal{W}_{(3^k)} \cong \mathcal{V}_{(3^k)}$$

as $\mathcal{U}(\mathfrak{sl}_n)$ 2-representations.

This will imply that

$$K_0(\mathcal{W}_{(3^k)}) \cong V_{(3^k)}^{\mathbb{Z}}.$$

Note that

$$K_0(\mathcal{W}_{(3^k)}) \cong \bigoplus_{\mu_S \in \Lambda(n, n)_3} K_0(K_S).$$

We will show that this corresponds exactly to the $U_q(\mathfrak{gl}_n)$ -weight space decomposition of $V_{(3^k)}$. In particular, this will show that

$$(5.3.1) \quad K_0(K_S) \cong W_S,$$

for any enhanced sign sequence S such that $\mu_S \in \Lambda(n, n)_3$.

First, we have to recall the definitions of *sweet* bimodules.

5.3.2. *Sweet bimodules.* Note that the following definitions and results are the \mathfrak{sl}_3 -analogues of those in Section 2.7 in [26].

Definition 5.3.2. Given rings R_1 and R_2 , a (R_1, R_2) -bimodule N is called *sweet* if it is finitely generated and projective as a left R_1 -module and as a right R_2 -module.

If N is a sweet (R_1, R_2) -bimodule, then the functor

$$N \otimes_{R_2} - : R_2\text{-mod} \rightarrow R_1\text{-mod}$$

is exact and sends projective modules to projective modules. Given a sweet (R_1, R_2) -bimodule M and a sweet (R_2, R_3) -bimodule N , then the tensor product $M \otimes_{R_2} N$ is a sweet (R_1, R_3) -bimodule.

Let S and S' be two enhanced sign strings. Then $\widehat{B}_S^{S'}$ denotes the set of all webs whose boundary is divided into a lower part, determined by S , and an upper part, determined by S' . Here we mean one diagram when we say web, not a linear combination of diagrams. Let $B_S^{S'} \subset \widehat{B}_S^{S'}$ be the subset of non-elliptic webs.

For any $w \in \widehat{B}_S^{S'}$, define a graded finite-dimensional $(K_{S'}, K_S)$ -bimodule $\Gamma(w)$ by

$$\Gamma(w) := \bigoplus_{u \in B_{S'}, v \in B_S} {}_u\Gamma(w)_v,$$

with

$${}_u\Gamma(w)_v := \mathcal{F}^c(u^*wv)\{n\},$$

where n is the length of S' . The left and right actions of K_S on $\Gamma(w)$ are defined by applying the multiplication foam in 3.0.18 to

$${}_rK_u \otimes {}_u\Gamma(w)_v \rightarrow {}_r\Gamma(w)_v \quad \text{and} \quad {}_u\Gamma(w)_v \otimes {}_vK_r \rightarrow {}_u\Gamma(w)_r.$$

Let $w \in \widehat{B}_S^{S'}$. Then $w = c_1w_1 + \cdots + c_mw_m$, for certain $w_i \in B_S^{S'}$ and $c_i \in \mathbb{N}[q, q^{-1}]$. Since all relations which are satisfied by the Kuperberg bracket have categorical analogues for foams, this shows that

$$\Gamma(w) \cong c_1\Gamma(w_1) \oplus \cdots \oplus c_m\Gamma(w_m),$$

where the multiplication by the c_i is interpreted in the usual way using direct sums and grading shifts.

We have the following analogue of Proposition 3 in [26].

Proposition 5.3.3. *For any $w \in \widehat{B}_S^{S'}$, the graded $(K_{S'}, K_S)$ -bimodule $\Gamma(w)$ is sweet.*

Proof. As a left K_S -module, we have

$$\Gamma(w) \cong \bigoplus_{v \in B_S} \Gamma(w)_v,$$

where

$$\Gamma(w)_v = \bigoplus_{u \in B_{S'}} {}_u\Gamma(w)_v.$$

So, as far as the left action is concerned, it suffices to show that $\Gamma(w)_v$ is a left projective $K_{S'}$ -module. Note that, as a left $K_{S'}$ -module, we have

$$\Gamma(w)_v \cong \bigoplus_{u \in B_{S'}} \mathcal{F}^0(wv).$$

Then $wv = c_1u_1 + \cdots + c_mu_m$, for certain $u_i \in B_{S'}$ and $c_i \in \mathbb{N}[q, q^{-1}]$. By the remarks above, this means that

$$\mathcal{F}^0(wv) \cong c_1P_{u_1} \oplus \cdots \oplus c_mP_{u_m},$$

which proves that $\Gamma(w)$ is projective as a left $K_{S'}$ -module.

The proof that $\Gamma(w)$ is projective as a right K_S -module is similar. □

It is not hard to see that (see for example [26]), for any $w \in \widehat{B}_S^{S'}$ and $w' \in \widehat{B}_{S'}^{S''}$, we have

$$(5.3.2) \quad \Gamma(ww') \cong \Gamma(w) \otimes_{K_{S'}} \Gamma(w').$$

Lemma 5.3.4. *Let $w, w' \in \widehat{B}_S^{S'}$. An isotopy between w and w' induces an isomorphism between $\Gamma(w)$ and $\Gamma(w')$. Two isotopies between w and w' induce the same isomorphism if and only if they induce the same bijection between the connected components of w and w' .*

Lemma 5.3.5. *Let $w, w' \in \widehat{B}_S^{S'}$ and let $f \in \mathbf{Foam}_3^0(w, w')$ be a foam of degree t . Then f induces a bimodule map*

$$\Gamma(f): \Gamma(w) \rightarrow \Gamma(w')$$

of degree t .

Proof. Note that, for any $u \in B_{S'}$ and $v \in B_S$, the foam f induces a linear map

$$\mathcal{F}^0(1_u^* f 1_v): \mathcal{F}^0(u^* w v) \rightarrow \mathcal{F}^0(u^* w' v),$$

by glueing $1_u^* f 1_v$ on top of any element in $\mathcal{F}^0(u^* w v) = \mathbf{Foam}_3^0(\emptyset, u^* w v)$. This map has degree t , e.g. the identity has degree 0 because the multiplication in K_S is degree preserving. By taking the direct sum over all $u \in B_{S'}$ and $v \in B_S$, we get a linear map

$$\Gamma(f): \Gamma(w) \rightarrow \Gamma(w').$$

The shifts in the definition of $\Gamma(w)$ and $\Gamma(w')$, given by the length n of w and the length m of w' , imply that $\deg \Gamma(f) = t$.

The fact that $\Gamma(f)$ is a left K_S -module map follows from the following observation. For any $u \in B_S$ and $v \in B_{S'}$, the linear map $\mathcal{F}^0(1_u^* f 1_v)$ corresponds to the linear map

$$\mathbf{Foam}_3^0(u, wv) \rightarrow \mathbf{Foam}_3^0(u, w'v)$$

determined by horizontally composing with $f 1_v$ on the right-hand side. This map clearly commutes with any composition on the left-hand side.

Analogously, the linear map $\mathcal{F}^0(1_u^* f 1_v)$ corresponds to the linear map

$$\mathbf{Foam}_3^0(w^* u, v) \rightarrow \mathbf{Foam}_3^0((w')^* u, v)$$

determined by horizontally composing with $f^* 1_u$ on the left-hand side. This map clearly commutes with any composition on the right-hand side.

These two observations show that $\Gamma(f)$ is a $(K_{S'}, K_S)$ -bimodule map. \square

It is not hard to see that, for any $f \in \mathbf{Foam}_3^0(w, w')$ and $g \in \mathbf{Foam}_3^0(w', w'')$, we have

$$\Gamma(fg) = \Gamma(f)\Gamma(g).$$

Similarly, for any $u_1, u_2 \in \widehat{B}_S^{S'}$ and $u'_1, u'_2 \in \widehat{B}_{S'}^{S''}$ and any $f \in \mathbf{Foam}_3^0(u_1, u_2)$ and $f' \in \mathbf{Foam}_3^0(u'_1, u'_2)$, we have a commuting square

$$\begin{array}{ccc} \Gamma(u_1 u'_1) & \xrightarrow{\Gamma(f \circ f')} & \Gamma(u_2 u'_2) \\ \cong \downarrow & & \cong \downarrow \\ \Gamma(u_1) \otimes_{K_{S'}} \Gamma(u'_1) & \xrightarrow{\Gamma(f) \otimes \Gamma(f')} & \Gamma(u_2) \otimes_{K_{S'}} \Gamma(u'_2) \end{array}$$

where the vertical isomorphisms are as in (5.3.2).

5.3.3. *The categorical $\mathcal{S}(n, n)$ -action on $\mathcal{W}_{(3^k)}$.* We are now going to use sweet bimodules to define a categorical action of $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$. For the definition of this action, we will consider $\mathcal{S}(n, n)$ to be a monoidal category rather than a 2-category.

Definition 5.3.6. On objects: the categorical action of any object $\mathcal{E}_i 1_\lambda$ in $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$ is defined by tensoring with the sweet bimodule (see Proposition 5.3.3)

$$\Gamma \left(\phi \left(\mathcal{E}_i 1_\lambda \right) \right).$$

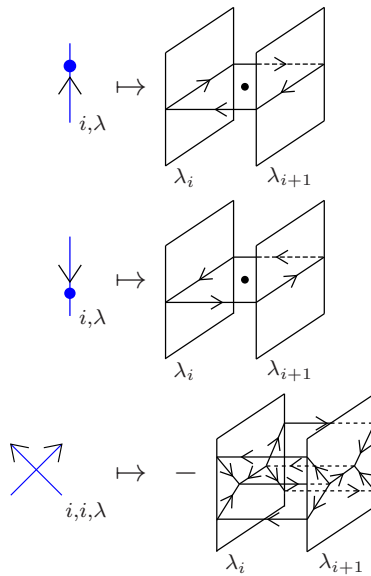
Recall that $\phi: \mathcal{S}_q(n, n) \rightarrow \text{End}_{\mathbb{C}(q)}(W_{(3^k)})$ was defined in Definition 5.2.2.

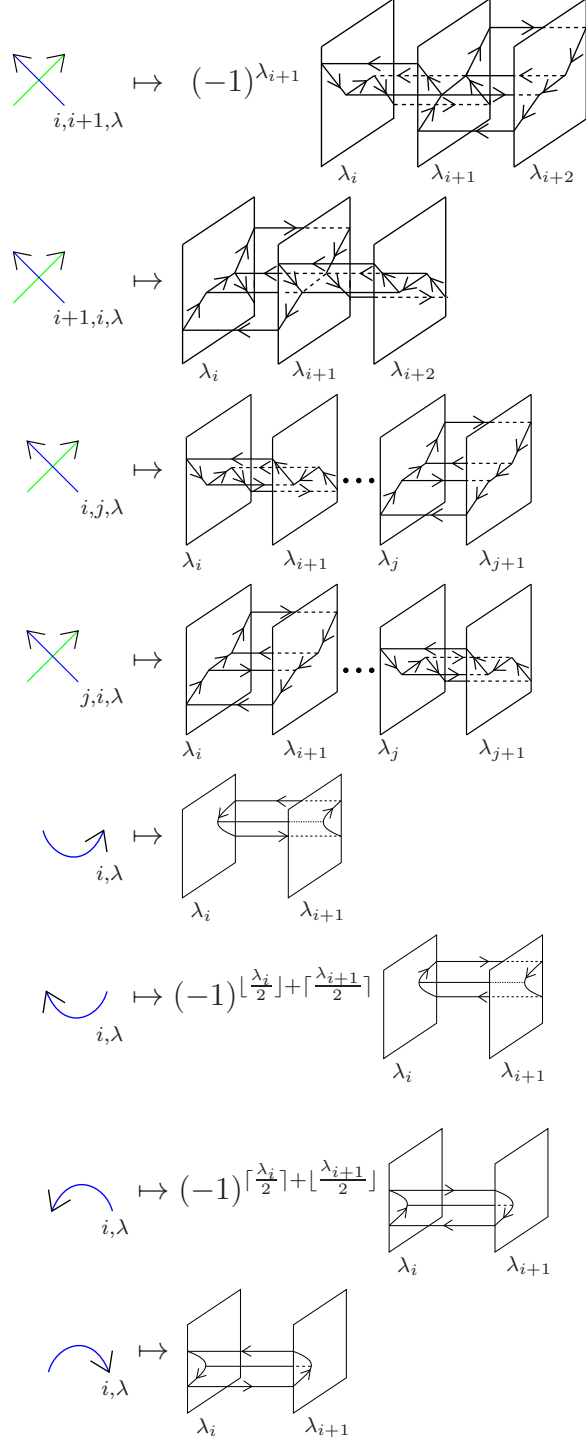
On morphisms: we give a list of the foams associated to the generating morphisms of $\mathcal{S}(n, n)$. Applying Γ to these foams determines the natural transformations associated to the morphisms of $\mathcal{S}(n, n)$.

As before, we only draw the most important part of the foams, omitting partial identity foams. Note our conventions:

- (1) We read the regions of the morphisms in $\mathcal{S}(n, n)$ from right to left and the morphisms themselves from bottom to top.
- (2) The corresponding foams we read from bottom to top and from front to back.
- (3) Vertical front edges labeled 1 are assumed to be oriented upward and vertical front edges labeled 2 are assumed to be oriented downward.
- (4) The convention for the orientation of the back edges is precisely the opposite.
- (5) A facet is labeled 0 or 3 if and only if its boundary has edges labeled 0 or 3.

In the list below, we always assume that $i < j$. Finally, all facets labeled 0 or 3 in the images below have to be erased, in order to get real foams. For any $\lambda > (3^k)$, the image of the elementary morphisms below is taken to be zero, by convention.





Proposition 5.3.7. *The formulas in Definition 5.3.6 determine a well-defined graded categorical action of $\mathcal{S}(n, n)$ on $\mathcal{W}_{(3^k)}$.*

Proof. A tedious but straightforward case by case check, for each generating morphism and each λ which give a non-zero foam, shows that each of the foams in Definition 5.3.6 has the same degree as the elementary morphism in $\mathcal{S}(n, n)$ to which it is associated. Note that it is important to erase the facets labeled 0 or 3, before computing the degree of the foams. We do just one example here.

We have

$$\begin{array}{c} \curvearrowright \\ i, (12) \end{array} \mapsto - \begin{array}{c} \text{foam } f \\ \text{with facets } 0, 2, 3 \\ \text{and edges } 1, 2 \end{array} \quad \text{and} \quad \deg \left(\begin{array}{c} \curvearrowright \\ i, (12) \end{array} \right) = 2.$$

We see that f has one facet labeled 0 and another labeled 3, so those two facets have to be erased. Therefore, f has 12 vertices, 14 edges and 3 faces, i.e.

$$\chi(f) = 12 - 14 + 3 = 1.$$

The boundary of f has 12 vertices and 12 edges, so

$$\chi(\partial f) = 12 - 12 = 0.$$

Note that the two circular edges do not belong to ∂f , because the circular facets have been removed. In this section we draw the foams horizontally, so b is the number of horizontal edges at the top and the bottom of f , which go from the front to the back. Thus, for f we have

$$b = 4.$$

Altogether, we get

$$q(f) = 0 - 2 + 4 = 2.$$

In order to show that the categorical action is well-defined, one has to check that it preserves all the relations in Definition 2.3.9. Modulo 2 this was done in the proof of Theorem 4.2 in [40]. At the time there was a small issue about the signs in [31], which prevented the author to formulate and prove Theorem 4.2 in [40] over \mathbb{C} . That issue has now been solved (see [41] and [32] for more information) and in this paper we use the sign conventions from [41], which are compatible with those from [32]. We laboriously checked all these relations again, but now over \mathbb{C} and with the signs above. The arguments are exactly the same, so let us not repeat them one by one here. Instead, we first explain how we computed the signs for the categorical action above and why they give the desired result over \mathbb{C} . After that, we will do an example. For a complete case by case check, we refer to the arguments used in the proof of Theorem 4.2 in [40]. The reader should check that our signs above remove the sign ambiguities in that proof.

One can compute the signs above as follows: first check the relations only involving strands of one color, i.e. the \mathfrak{sl}_2 -relations. The first thing to notice is that the foams in the categorical action do not satisfy relation (2.3.20); for all λ , which give a non-zero foam, the sign is wrong. Therefore, one is forced to multiply the foam associated to

$$\begin{array}{c} \text{foam} \\ i, i, \lambda \end{array}$$

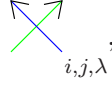
by -1 , for all λ .

After that, compute the foams associated to the degree zero bubbles (real bubbles, not fake bubbles) and adjust the signs of the images of the left cups and caps accordingly. This way, most of the signs of the images of the left cups and caps get determined. The remaining ones can be determined by imposing the zig-zag relations in (2.3.8) and (2.3.9).

Of course, one could also choose to adjust the signs of the images of the right cups and caps. That would determine a categorical action that is naturally isomorphic to the one in this paper.

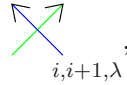
After these signs have been determined, one can check that all \mathfrak{sl}_2 -relations are preserved by the categorical action.

The next and final step consists in determining the signs of



for $i \neq j$. First one can check that cyclicity is already preserved. The relations in (2.3.11) are preserved by the corresponding foams, which are all isotopic, with our sign choices for the foams associated to the left cups and caps. Therefore, cyclicity does not determine any more signs.

The relations in (2.3.21) are preserved on the nose, for $i = j$ and $|i - j| > 1$. For $|i - j| = 1$, they are only preserved up to a sign. Note that, since the corresponding foams are all isotopic, the signs actually come from the sign choice for the foams associated to the left cups and caps. Thus, whenever the total sign in the image of (2.3.21) becomes negative, one has to change the sign of one of the two crossings (not of both of course). Our choice has been to change the sign of the foam associated to

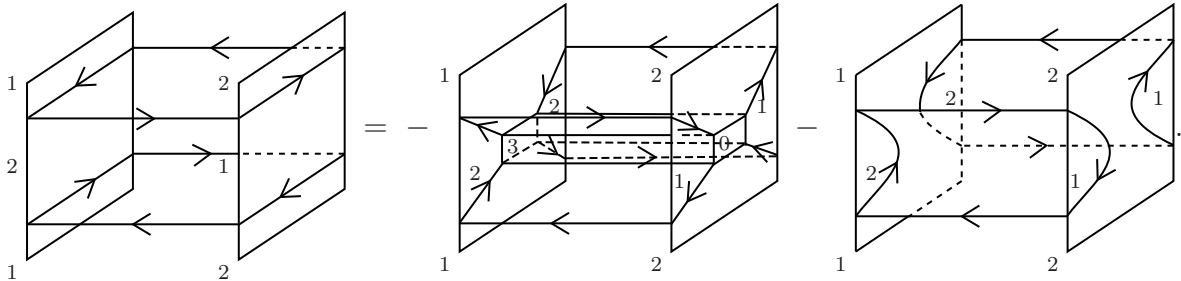


whenever necessary. Any other choice, consistent with all the previous sign choices, leads to a naturally isomorphic categorical action. It turns out that the sign has to be equal to $(-1)^{\lambda_{i+1}}$, after checking for all λ .

After this, one can check that all relations involving two or three colors are preserved by the categorical action. Note that we have not specified an image for the fake bubbles. As stressed repeatedly in [31], fake bubbles do not exist as separate entities. They are merely formal symbols, used as computational devices to keep the computations involving real bubbles tidy and short. As we are using \mathfrak{sl}_3 -foams in this paper, most of the dotted bubbles are mapped to zero. Therefore, under the categorical action it is very easy to convert the fake bubbles in the relations in Definition 2.3.9 into linear combinations of real bubbles, using the infinite Grassmannian relation (2.3.18). Thus, there is no need to use fake bubbles in this paper.

Finally, let us do two examples; one involving only one color and another involving two colors.

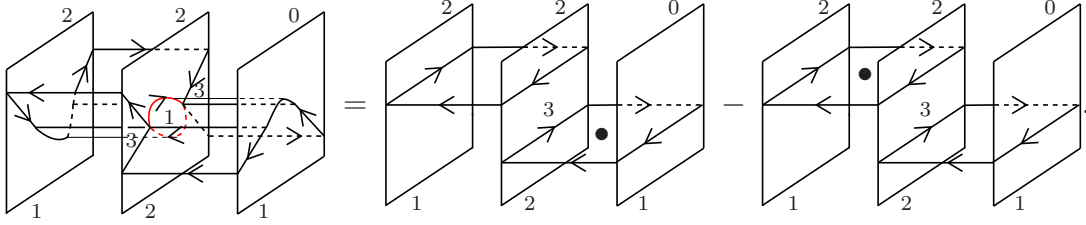
The left side of the equation in (2.3.17), for $i = 1$ and $\lambda = (1, 2)$ (the other entries are omitted for simplicity), becomes



This foam equation is precisely the relation (SqR). Note that the signs match perfectly, because we have

$$\text{sign} \left(\begin{array}{c} \text{blue arc} \\ i, (12) \end{array} \right) = + \quad \text{and} \quad \text{sign} \left(\begin{array}{c} \text{green arc} \\ i, (12) \end{array} \right) = -.$$

The equation in (2.3.22), for $(i, j) = (1, 2)$ and $\lambda = (121)$ (the other entries are omitted for simplicity), becomes



To see that this holds, apply the (RD) relation to the foam on the l.h.s., in order to remove the disc bounded by the red singular circle on the middle sheet. \square

Let $W_h \cong \mathbb{C}$ be the unique indecomposable projective graded $K_{(\times^k, \circ^{2k})}$ -module of degree zero. Recall that $K_{(\times^k, \circ^{2k})}$ is generated by the empty diagram, so W_h is indeed one-dimensional. It is the categorification of w_h , the highest weight vector in $W_{(3^k)}$.

Note that we can pull back the categorical action on $\mathcal{W}_{(3^k)}$ via

$$\Psi_{n,n}: \mathcal{U}(\mathfrak{sl}_n) \rightarrow \mathcal{S}(n, n).$$

We are now able to prove one of our main results. Recall that \mathcal{V} is any additive idempotent complete category, which allows an integrable graded categorical action by $\mathcal{U}(\mathfrak{sl}_n)$.

Theorem 5.3.8. *There exists an equivalence of categorical $\mathcal{U}(\mathfrak{sl}_n)$ -representations*

$$\Phi: \mathcal{V}_{(3^k)} \rightarrow \mathcal{W}_{(3^k)}.$$

Proof. As we already mentioned above, we have

$$\text{End}_{\mathcal{W}_{(3^k)}}(W_h) \cong \mathbb{C}.$$

Let Q be any indecomposable object in $\mathcal{W}_{(3^k)}$. There exists an enhanced sign string S such that Q belongs to $K_S\text{-pmod}_{\text{gr}}$. Therefore, there exists a basis web $w \in B_S$ and a $t \in \mathbb{Z}$, such that Q is a graded direct summand of $P_w\{t\}$. Without loss of generality, we may assume that $t = 0$.

By Lemma 5.2.10 and Proposition 5.3.7, there exists an object of X in $\mathcal{S}(n, n)$ such that Q is a direct summand of XW_h . This holds, because in $\dot{\mathcal{S}}(n, n)$, the Karoubi envelope of $\mathcal{S}(n, n)$, the divided powers correspond to direct summands of ordinary powers. For more details on the categorification of the divided powers see [31] and [33].

Proposition 2.3.15 now proves the existence of Φ . \square

An easy consequence of Theorem 5.3.8 is the following:

Corollary 5.3.9. *By Theorem 5.3.8, the $S_q^{\mathbb{Z}}(n, n)$ -module map*

$$K_0(\Phi): K_0(\mathcal{V}_{(3^k)}) \rightarrow K_0(\mathcal{W}_{(3^k)})$$

is an isomorphism.

The following consequence of Theorem 5.3.8 is very important and we thank Ben Webster for explaining its proof.

Proposition 5.3.10. *The graded algebras $K_{(3^k)}$ and $R_{(3^k)}$ are Morita equivalent.*

Proof. We are going to show that, for each weight μ_S which shows up in the weight decomposition of $V_{(3^k)}$, the graded algebras K_S and $R(\mu_S - \lambda, \lambda)$ are Morita equivalent. This proves the proposition after taking direct sums.

Let μ_S be a weight which shows up in the weight decomposition of $V_{(3^k)}$. Define

$$\Theta_{\mu_S} := \bigoplus_{\mathbf{i} \in \text{Seq}(\mu_S - \lambda)} \mathcal{E}_{\mathbf{i}} W_h \in K_S\text{-pmod}_{\text{gr}}.$$

In the proof of Theorem 5.3.8, we already showed that every object in $K_S\text{-pmod}_{\text{gr}}$ is a direct summand of XW_h for some object $X \in \mathcal{S}(n, n)$. By the biadjointness of the \mathcal{E}_i and \mathcal{E}_{-i} in $\mathcal{S}(n, n)$ and the fact that W_h is a highest weight object, it is not hard to see that XW_h itself is a direct summand of a finite direct sum of degree shifted copies of Θ_{μ_S} . This shows that every object in $K_S\text{-pmod}_{\text{gr}}$ is a direct summand of a finite direct sum of degree shifted copies of Θ_{μ_S} . Since K_S is a finite-dimensional complex algebra, every graded finite-dimensional K_S -module has a projective cover and is therefore a quotient of a finite direct sum of degree-shifted copies of Θ_{μ_S} . This shows that Θ_{μ_S} is a projective generator of $K_S\text{-mod}_{\text{gr}}$.

Theorem 5.3.8 also shows that

$$\text{End}_{K_S}(\Theta_{\mu_S}) \cong R(\mu_S - \lambda, \lambda)$$

holds.

By a general result due to Morita, it follows that the above observations imply that K_S and $R(\mu_S - \lambda, \lambda)$ are Morita equivalent. For a proof see Theorem 5.55 in [50], for example. \square

We can draw two interesting conclusions from Proposition 5.3.10.

In [7], Brundan and Kleshchev defined an explicit isomorphism between blocks of cyclotomic Hecke algebras and cyclotomic KLR-algebras. Theorem 3.2 in [5] implies that the center of the cyclotomic Hecke algebra, which under Brundan and Kleshchev's isomorphism corresponds to $R(\mu_S - \lambda, \lambda)$, has the same dimension as $H^*(X_{\mu_S}^{(3^k)})$.

Corollary 5.3.11. *The center of K_S is isomorphic to the center of $R(\mu_S - \lambda, \lambda)$. In particular, we have*

$$\dim Z(K_S) = \dim Z(R(\mu_S - \lambda, \lambda)) = \dim H^*(X_{\mu_S}^{(3^k)}).$$

Proof. We only have to prove the first statement, which follows from the well known fact that Morita equivalent algebras have isomorphic centers. For a proof see for example Corollary 18.42 in [37]. \square

In Theorem 4.2.3 we used Corollary 5.3.11 to give an explicit isomorphism

$$H^*(X_{\mu_S}^{(3^k)}) \rightarrow Z(K_S).$$

Remark 5.3.12. Just for completeness, we remark that the aforementioned results in [7] and [5] together with the results in [4], which we have not explained, imply that

$$H^*(X_{\mu_S}^{(3^k)}) \cong Z(R(\mu_S - \lambda, \lambda)),$$

so we have not proved anything new about $Z(R(\mu_S - \lambda, \lambda))$.

Another interesting consequence of Proposition 5.3.10 is the following:

Corollary 5.3.13. *K_S is a graded cellular algebra.*

Proof. In Corollary 5.12 in [45], Hu and Mathas proved that $R(\mu_S - \lambda, \lambda)$ is a graded cellular algebra.

In [35], König and Xi showed that “being a cellular algebra” is a Morita invariant property, provided that the algebra is defined over a field whose characteristic is not equal to two.

These two results together with Proposition 5.3.10 prove that K_S is a graded cellular algebra. \square

The precise definition of a graded cellular algebra can be found in [45]. We will not recall it here. In a follow-up paper, we intend to discuss the cellular basis of K_S in detail and use it to derive further results on the representation theory of K_S .

Remark 5.3.14. Corollary 5.3.13 is the \mathfrak{sl}_3 analogue of Corollary 3.3 in [9], which proves that Khovanov’s arc algebra H_m is a graded cellular algebra. It is easy to give a cellular basis of H_m . The proof of cellularity follows from checking a small number of cases by hand. For K_S , we tried to mimick that approach, but had to give up because the combinatorics got too complex.

5.3.4. The Grothendieck group of $W_{(3^k)}$. Recall that $W_S^{\mathbb{Z}}$ has an inner product defined by the normalized Kuperberg form (see Definition 5.2.7). The Euler form

$$\langle [P], [Q] \rangle := \dim_q \text{HOM}(P, Q)$$

defines a $\mathbb{Z}[q, q^{-1}]$ sesquilinear form on $K_0(K_S)$.

Lemma 5.3.15. *Let S be an enhanced sign sequence. Take*

$$\gamma_S: W_S^{\mathbb{Z}} \rightarrow K_0(K_S)$$

to be the $\mathbb{Z}[q, q^{-1}]$ linear map defined by

$$\gamma_S(u) = [P_u],$$

for any $u \in B_S$. Then γ_S is an isometric embedding.

This implies that the $\mathbb{Z}[q, q^{-1}]$ linear map

$$\gamma_W := \bigoplus_{\mu(S): \Lambda(n,n)_3} \gamma_S$$

defines an isometric embedding

$$\gamma_W: W_{(3^k)}^{\mathbb{Z}} \rightarrow K_0(W_{(3^k)}).$$

Proof. Note that the normalized Kuperberg form, because of the relations 2.1.3, 2.1.4 and 2.1.5, and the Euler form are non-degenerate. For any pair $u, v \in B_S$, we have

$$\dim_q \text{HOM}(P_u, P_v) = \dim_q {}_u K_v = q^{\ell(\hat{S})} \langle u^* v \rangle.$$

The factor $q^{\ell(\hat{S})}$ is a consequence of the grading shift in the definition of ${}_u K_v$.

Thus, γ_S is an isometry. Since the normalized Kuperberg form is non-degenerate, this implies that γ_S is an embedding. \square

Remark 5.3.16. It is well-known that $K_0(K_S)$ is the free $\mathbb{Z}[q, q^{-1}]$ module generated by the isomorphism classes of the indecomposable projective K_S modules. In Section 5.5 in [47], Morrison and Nieh showed that P_u is not necessarily indecomposable (see also [48]). This is closely related to the contents of Remark 2.1.6, as Morrison and Nieh showed. Therefore, the surjectivity of γ_W is not immediately clear and we need the results of the previous subsections to establish it below.

The \mathfrak{sl}_2 case is much simpler. The projective modules analogous to the P_u are all indecomposable. See Proposition 2 in [26] for the details.

Theorem 5.3.17. *The map*

$$\gamma_W: W_{(3^k)}^{\mathbb{Z}} \rightarrow K_0(\mathcal{W}_{(3^k)})$$

is an isomorphism of $S_q^{\mathbb{Z}}(n, n)$ -modules.

This also implies that, for each sign string S , the map

$$\gamma_S: W_S^{\mathbb{Z}} \rightarrow K_0(K_S)$$

is an isomorphism.

Proof. The proof of the theorem is only a matter of assembling already known pieces.

By Proposition 5.3.7, γ_W intertwines the $S_q^{\mathbb{Z}}(n, n) \cong K_0(\dot{S}(n, n))$ actions.

We already know that γ_W is an embedding, by Lemma 5.3.15.

Note that, by Theorem 2.3.14, Lemma 5.2.6 and Corollary 5.3.9, we have the following commuting square

$$\begin{array}{ccc} V_{(3^k)}^{\mathbb{Z}} & \xrightarrow{\gamma_V} & K_0(\mathcal{V}_{(3^k)}) \\ \phi \downarrow & & K_0(\Phi) \downarrow \\ W_{(3^k)}^{\mathbb{Z}} & \xrightarrow{\gamma_W} & K_0(\mathcal{W}_{(3^k)}). \end{array}$$

We already know that γ_V , ϕ and $K_0(\Phi)$ are isomorphisms. Therefore, γ_W has to be an isomorphism. This shows that K_S indeed categorifies the μ_S -weight space of $V_{(3^k)}$.

Recall that we have not explained the definition of γ_V nor Rouquier's definition of Φ . However, for general reasons, γ_V has to send the highest weight vector $v_h \in V_{(3^k)}^{\mathbb{Z}}$ to the class of the highest weight object in $\mathcal{V}_{(3^k)}$ and Φ has to send that highest weight object to the highest weight object in $\mathcal{W}_{(3^k)}$. This shows that the images of the highest weight vector $v_h \in V_{(3^k)}^{\mathbb{Z}}$ around the two sides of the square are equal. Since all maps involved are $S_q^{\mathbb{Z}}(n, n)$ intertwiners, it follows that the square indeed commutes. \square

A good question is how to find the indecomposable graded projective modules of K_S . Before answering that question, we need a result on the 3-colorings of webs.

Let $w \in B_S$. Recall that there is a bijection between the flows on w and the 3-colorings of w , as already mentioned in Remark 2.1.2. Call the 3-coloring corresponding to the canonical flow of w , the *canonical 3-coloring*, denoted T_w .

Lemma 5.3.18. *Let $u, v \in B_S$. If there is a 3-coloring of v which matches T_u and a 3-coloring of u which matches T_v on the common boundary S , then $u = v$.*

Proof. This result is a direct consequence of Theorem 2.1.5. Recall that there is a partial order on flows, and therefore on 3-colorings by Remark 2.1.2. This ordering is induced by the lexicographical order on the state-strings on S , which are induced by the flows. Note that two matching colorings of u and v have the same order, by definition. On the other hand, Theorem 2.1.5 implies that any 3-coloring of u , respectively v , has order less than or equal to that of T_u and T_v respectively. Therefore, if there exists a 3-coloring of v matching T_u , the order of T_u must be less or equal than that of T_v .

Thus, if there exists a 3-coloring of v matching T_u and a 3-coloring of u matching T_v , then T_u and T_v must have the same order. This implies that $u = v$, because canonical 3-colorings are uniquely determined by their order and the corresponding canonical flows determine the corresponding basis webs uniquely by the growth algorithm. \square

Proposition 5.3.19. *For each $u \in B_S$, there exists a unique graded indecomposable projective K_S -module Q_u , such that*

$$P_u \cong Q_u \oplus \bigoplus_{J_v < J_u} d(S, J_u, J_v) Q_v.$$

Here J_u is the state string associated to the canonical flow on u , the coefficients $d(S, J_u, J_v)$ belong to $\mathbb{N}[q, q^{-1}]$ and indicate direct sums and degree shifts as usual, and the state strings are ordered lexicographically.

Note that we need a lot of the results from the Sections 3, 4 and 5 to prove the proposition.

Proof. Let $u \in B_S$. Then there is a complete decomposition of 1_u into orthogonal primitive idempotents

$$1_u = e_1 + \cdots + e_r.$$

By Theorem A.0.26 and Corollary A.0.27, we can lift this decomposition to G_S . We do not introduce any new notation for this lift, trusting that the reader will not get confused by this slight abuse of notation.

Let $z_u \in Z(G_S)$ be the central idempotent corresponding to J_u , as defined in the proof of Lemma 4.2.2. We claim that there is a unique $1 \leq i \leq r$, such that

$$(5.3.3) \quad z_u e_j = \delta_{ij} z_u 1_u,$$

for any $1 \leq j \leq r$.

Let us prove this claim. Note that

$$(5.3.4) \quad z_u 1_u = e_{u, T_u},$$

where T_u is the canonical coloring of u , i.e. J_u only allows one compatible coloring of u , which is T_u . Since $e_{u, T_u} \neq 0$, courtesy of Lemma 3.0.26, this implies that

$$(5.3.5) \quad z_{uu} G_u = {}_u G_u z_u = z_{uu} G_u z_u = e_{u, T_u} G_S e_{u, T_u} \cong \mathbb{C},$$

by Theorem 3.0.25.

We also see that there has to exist at least one $1 \leq i_0 \leq r$ such that $z_u e_{i_0} \neq 0$. Then, by (5.3.5), there exists a non-zero $\lambda_{i_0} \in \mathbb{C}$, such that

$$z_u e_{i_0} = \lambda_{i_0} z_u 1_u = \lambda_{i_0} e_{u, T_u}.$$

For any $1 \leq i, j \leq r$, we have

$$z_u e_i z_u e_j = z_u^2 e_i e_j = z_u \delta_{ij} e_i.$$

This implies that i_0 is unique and $\lambda_{i_0} = 1$. In order to see that this is true, suppose there exist $1 \leq i_0 \neq j_0 \leq r$ such that $z_u e_{i_0} \neq 0$ and $z_u e_{j_0} \neq 0$. By (5.3.5), there exist non-zero $\lambda_{i_0}, \lambda_{j_0} \in \mathbb{C}$ such that

$$z_u e_{i_0} = \lambda_{i_0} z_u 1_u \quad \text{and} \quad z_u e_{j_0} = \lambda_{j_0} z_u 1_u.$$

However, this is impossible, because we get

$$z_u e_{i_0} z_u e_{j_0} = \lambda_{i_0} \lambda_{j_0} z_u 1_u \neq 0,$$

which contradicts the orthogonality of $z_u e_{i_0}$ and $z_u e_{j_0}$.

Thus, for each $u \in B_S$, there is a unique primitive idempotent $e_u \in \text{End}_{\mathbb{C}}(P_u)$ that is not killed by z_u , when lifted to G_S . We define Q_u to be the corresponding graded indecomposable projective K_S -module:

$$Q_u := K_S e_u,$$

which is clearly a direct summand of $P_u = (K_S)1_u$.

Let us now show that, for any $u, v \in B_S$, we have

$$Q_u \cong Q_v \Leftrightarrow u = v.$$

If $u = v$, we obviously have $Q_u \cong Q_v$. Let us prove the other implication. Suppose $Q_u \cong Q_v$. From the above, recall that e_u and e_v can be lifted to G_S . By a slight abuse of notation, call these lifted idempotents e_u and e_v again. We have

$$z_u e_u = e_{u, T_u} \neq 0 \quad \text{and} \quad z_v e_v = e_{v, T_v} \neq 0.$$

Since $Q_u \cong Q_v$, we then also have

$$z_u e_v \neq 0 \quad \text{and} \quad z_v e_u \neq 0.$$

This can only hold if T_u gives a 3-coloring of v and T_v a 3-coloring of u . By Lemma 5.3.18, this implies that $u = v$.

Since

$$\text{rk}_{\mathbb{Z}[q, q^{-1}]} K_0(K_S) = \text{rk}_{\mathbb{Z}[q, q^{-1}]} W_S^{\mathbb{Z}} = \#B_S,$$

by Theorem 5.3.17, the above shows that

$$\{Q_u \mid u \in B_S\}$$

is a basis of the free $\mathbb{Z}[q, q^{-1}]$ module $K_0(K_S)$. For any $u, v \in B_S$, we have

$$z_u 1_u = z_u e_u \quad \text{and} \quad z_v 1_u = 0, \text{ if } J_v > J_u.$$

The second claim follows from the fact that there are no admissible 3-colorings of u greater than J_u . The proposition now follows. \square

Remark 5.3.20. Proposition 5.3.19 proves the conjecture about the decomposition of 1_u , which Morrison and Nieh formulate in the text between Conjectures 5.14 and 5.15 in [47].

Before giving the last result of this section, we briefly recall some facts about the *dual canonical basis* of $W_S^{\mathbb{Z}}$. For more details see [19] and [29]. There exists a q antilinear involution $\tilde{\psi}$ on $V_S^{\mathbb{Z}}$ (in [19] and [29] this involution is denoted ψ' and Φ , respectively). For any sign string S and any state string J , there exists a unique element $e_{\heartsuit J}^S \in V_S^{\mathbb{Z}}$ which is invariant under $\tilde{\psi}$ and such that

$$(5.3.6) \quad e_{\heartsuit J}^S = e_J^S + \sum_{J' < J} c(S, J, J') e_{J'}^S,$$

with $c(S, J, J') \in q\mathbb{Z}[q]$. Note that $q = v^{-1}$ in [19] and [29]. The e_J^S are the elementary tensors, which were defined in 2.1.5. The basis $\{e_{\heartsuit J}^S\}$ is called the *dual canonical basis* of $V_S^{\mathbb{Z}}$. Restriction to the dominant closed paths (S, J) gives the dual canonical basis of $W_S^{\mathbb{Z}}$ (see Theorem 3 in [29] and the comments below it).

We have not given a definition of $\tilde{\psi}$, but we note that $\tilde{\psi}$ is completely determined by Proposition 2 in [29]:

Proposition 5.3.21 (Khovanov-Kuperberg). *Each basis web $w \in B_S$ is invariant under $\tilde{\psi}$.*

The above definition is hard to check directly for $\{[Q_u] \mid u \in B_S\}$. Therefore, let us recall Webster's [61] very general definition of a canonical basis of a free $\mathbb{Z}[q, q^{-1}]$ -module M . Our q corresponds to q^{-1} in [61]. A *pre-canonical structure* on M is a choice of

- a q antilinear “bar involution” $\psi: M \rightarrow M$,
- a sesquilinear inner product $\langle -, - \rangle: M \times M \rightarrow \mathbb{Z}((q))$, and
- a “standard basis” $\{a_c\}_{c \in C}$ with partially ordered index set $(C, <)$ such that

$$(5.3.7) \quad \psi(a_c) \in a_c + \sum_{c' < c} \mathbb{Z}[q, q^{-1}]a_{c'}.$$

A basis $\{b_c\}_{c \in C}$ is called *canonical* if

- (1) each vector b_c is invariant under ψ ,
- (2) each vector b_c belongs to $a_c + \sum_{c' < c} \mathbb{Z}[q, q^{-1}]a_{c'}$
- (3) the vectors b_c are *almost orthonormal* in the sense that

$$(5.3.8) \quad \langle b_c, b_{c'} \rangle \in \delta_{c,c'} + q\mathbb{Z}[q].$$

If a canonical basis exists, for a given pre-canonical structure, then it is unique by Theorem 26.3.1 in [39]. In particular, the dual canonical basis is “canonical” in the above sense, w.r.t. to a pre-canonical structure which we will discuss below. We note that the same basis can be canonical w.r.t. different pre-canonical structures.

Let us show how Lusztig's canonical basis on $V_{(3^k)}^{\mathbb{Z}} \cong K_0(\mathcal{V}_{(3^k)})$ is mapped to a basis in $W_{(3^k)}^{\mathbb{Z}} \cong K_0(\mathcal{W}_{(3^k)})$, which is also canonical according to Webster's definition. After doing that, we will prove that the latter basis is exactly the dual canonical basis defined in [17] and [29].

First the pre-canonical structures.

- As Webster shows in Proposition 1.2 in [61], the bar involution on $K_0(\mathcal{V}_{(3^k)})$ is induced by Khovanov and Lauda's [31] contravariant functor

$$\psi: R_{(3^k)} \rightarrow R_{(3^k)},$$

given by reflecting the diagrams in the x -axis and inverting their orientation. On objects this functor sends $\mathcal{F}_i\{t\}$ to $\mathcal{F}_i\{-t\}$.

Using our equivalence

$$\Phi: \mathcal{V}_{(3^k)} \rightarrow \mathcal{W}_{(3^k)}$$

from Theorem 5.3.8, we get a contravariant functor

$$\psi: \mathcal{W}_{(3^k)} \rightarrow \mathcal{W}_{(3^k)}$$

given by reflecting the foams in the vertical yz -plane, i.e. the plane parallel to the front and the back of the foams in Definition 5.3.6, and inverting the orientation of their edges.

We have

$$\psi(P_u) \cong P_u, \quad \text{for any non-elliptic web } u.$$

It might seem confusing that P_u is again a left and not a right $K_{(3^k)}$ -module. The reason is that any $f \in K_{(3^k)}$ acts on $\psi(P_u)$ by multiplication on the right with $\psi(f)$. Since ψ is contravariant, this gives a left action again.

Using the isomorphism

$$\psi: W_{(3^k)}^{\mathbb{Z}} \rightarrow K_0(\mathcal{W}_{(3^k)})$$

to pull back $K_0(\psi)$, we get a bar involution on $W_{(3^k)}^{\mathbb{Z}}$ which fixes the non-elliptic webs. By Proposition 5.3.21, we see that this bar involution is equal to $\tilde{\psi}$.

- As remarked by Webster in the introduction of [61], the inner product on $K_0(\mathcal{V}_{(3^k)})$ is given by the Euler form

$$\langle [P], [Q] \rangle = \dim_q (\text{HOM}(P, Q)).$$

Pulling back the Euler form via the isomorphism

$$\gamma: V_{(3^k)}^{\mathbb{Z}} \rightarrow K_0(\mathcal{V}_{(3^k)})$$

gives the q -Shapovalov form.

Our isomorphism

$$K_0(\Phi): K_0(\mathcal{V}_{(3^k)}) \rightarrow K_0(\mathcal{W}_{(3^k)})$$

from Theorem 5.3.17 is an isometry intertwining the Euler forms. Furthermore, the Euler form on the latter Grothendieck group corresponds to the normalized Kuperberg form on $W_{(3^k)}^{\mathbb{Z}}$, by Lemma 5.2.8.

- For our purpose, we are only interested in a standard basis on $K_0(\mathcal{W}_{(3^k)})$. We take $\{[P_u]\}$, where the u are the non-elliptic webs in $W_{(3^k)}^{\mathbb{Z}}$. The partial ordering is given by the lexicographical ordering of the state-strings for each S . By Proposition 5.3.21, we see that $\{[P_u]\}$ satisfies (5.3.7).

Now, let us have a look at the canonical bases in $K_0(\mathcal{V}_{(3^k)})$ and $\mathcal{W}_{(3^k)}$, which both satisfy Webster's definition.

- (1) The canonical basis elements in $K_0(\mathcal{V}_{(3^k)})$ are the classes of the indecomposable projective $R_{(3^k)}$ -modules, with their gradings normalized such that they are direct summands of $R_{(3^k)}$. These elements correspond precisely to Lusztig's canonical basis elements in $V_{(3^k)}^{\mathbb{Z}}$, as shown by Brundan and Kleshchev [6]. In particular, they satisfy the three conditions for a canonical basis in Webster's list.

Our equivalence in Theorem 5.3.8 maps the indecomposable objects in $\mathcal{V}_{(3^k)}$ to the indecomposable objects in $\mathcal{W}_{(3^k)}$, which we had called Q_u . In particular, this shows that

$$\tilde{\psi}([Q_u]) = [Q_u].$$

- (2) The $[Q_u]$ also satisfy the second condition in Webster's list, as follows from inverting the change of basis matrix in Proposition 5.3.19.
- (3) The third condition in Webster's list, for the $[Q_u]$, follows from the fact that Lusztig's canonical basis elements $[P_u]$ satisfy that condition and the fact that the isomorphism

$$K_0(\Phi): K_0(\mathcal{V}_{(3^k)}) \rightarrow K_0(\mathcal{W}_{(3^k)}),$$

with Φ the equivalence in Theorem 5.3.17, maps Lusztig's canonical basis isometrically onto $\{Q_u\}$.

Theorem 5.3.22. *The basis*

$$\{[Q_u] \mid u \in B_S\}$$

corresponds to the dual canonical basis of $\text{Inv}(V_S^{\mathbb{Z}})$, under the isomorphisms

$$\text{Inv}(V_S^{\mathbb{Z}}) \cong W_S^{\mathbb{Z}} \cong K_0(K_S).$$

Proof. The remarks above prove that $\{[Q_u] \mid u \in B_S\}$ is a canonical basis in the sense of Webster's definition. What remains to be proven, is that it is exactly the dual canonical basis defined by Frenkel, Khovanov and Kirillov Jr. [19] (and in Theorem 3 in [29]).

As we demonstrated above, the bar involution on $K_0(\mathcal{W}_{(3^k)})$ is exactly the bar involution for the dual canonical basis in [19] and [29].

As we will explain below, the normalized Kuperberg $\mathbb{Z}[q, q^{-1}]$ sesquilinear form on $K_0(\mathcal{W}_{(3^k)})$, given in Definition 5.2.7 and denoted by $\langle -, - \rangle_{\text{Kup}}$ in this proof, is exactly the one corresponding to the pre-canonical structure used in [19] and [29].

Since there is at most one canonical basis for any given pre-canonical structure on $K_0(\mathcal{W}_{(3^k)})$, this proves that the two bases are equal.

For completeness, let us explain why $\langle -, - \rangle_{\text{Kup}}$ is exactly equal to the $\mathbb{Z}[q, q^{-1}]$ sesquilinear inner product that is used *implicitly* in [19] and [29]. The form that is used *explicitly* in [19] and [29] is actually Lusztig's $\mathbb{Z}[q, q^{-1}]$ bilinear form, denoted $(-, -)_{\text{Lus}}$ in this proof and defined in Section 19.1.1 in [39] for irreducible modules and extended factorwise to tensor products in Section 27.3 of that same book.

Therefore, we first have to recall how $\mathbb{Z}[q, q^{-1}]$ bilinear forms are related to $\mathbb{Z}[q, q^{-1}]$ sesquilinear forms. Given a $\mathbb{Z}[q, q^{-1}]$ bilinear form $(-, -)$ on $V_S^{\mathbb{Z}}$, we can define a $\mathbb{Z}[q, q^{-1}]$ sesquilinear form on $V_S^{\mathbb{Z}}$ which is $\mathbb{Z}[q, q^{-1}]$ antilinear in the first variable, by

$$(5.3.9) \quad \langle x, y \rangle := (\tilde{\psi}(x), y),$$

where $\tilde{\psi}$ is the $\mathbb{Z}[q, q^{-1}]$ antiinvolution mentioned above. This is exactly how Khovanov and Lauda defined their $\mathbb{Z}[q, q^{-1}]$ sesquilinear form on $\dot{\mathcal{U}}(\mathfrak{sl}_n)$ in Definition 2.3 in [31].

We do not compute the action of $\tilde{\psi}$ on the elementary tensors e_j^S explicitly in this paper. As we will show below, the e_j^S are orthonormal w.r.t. $(-, -)_{\text{Lus}}$. Therefore, it is easier to show that $(-, -)_{\text{Lus}}$ is equal to the $\mathbb{Z}[q, q^{-1}]$ bilinear form coming from Kuperberg's bracket, which we denote by $(-, -)_{\text{Kup}}$ in this proof, than to compare the corresponding $\mathbb{Z}[q, q^{-1}]$ sesquilinear forms directly. Just for the record, we remark that $\langle -, - \rangle_{\text{Lus}}$ is not equal to the factorwise q -Shapovalov form, which is part of the pre-canonical structure for Lusztig's canonical basis of $V_S^{\mathbb{Z}}$ (see Theorem 3.10 in [61]).

Let us recall the definition of $(-, -)_{\text{Lus}}$ on an irreducible weight $\dot{\mathcal{U}}^{\mathbb{Z}}(\mathfrak{sl}_3)$ module $V^{\mathbb{Z}}$ with highest weight vector v_h . We follow Khovanov and Lauda's normalization from Proposition 2.2 in [31]. Lusztig's $\mathbb{Z}[q, q^{-1}]$ bilinear form on $V^{\mathbb{Z}}$ is uniquely determined by the properties

- $(v_h, v_h)_{\text{Lus}} = 1$;
- $(ux, y)_{\text{Lus}} = (x, \bar{\rho}(u)y)_{\text{Lus}}$;
- $(y, x)_{\text{Lus}} = (x, y)_{\text{Lus}}$,

for any $x, y \in V^{\mathbb{Z}}$ and any $u \in \dot{\mathcal{U}}^{\mathbb{Z}}(\mathfrak{sl}_3)$. Here $\bar{\rho}$ is the $\mathbb{Z}[q, q^{-1}]$ linear antiinvolution on $\dot{\mathcal{U}}^{\mathbb{Z}}(\mathfrak{sl}_3)$ defined by

$$\rho(E_i) = q^{-1}K_i^{-1}E_{-i}, \quad \rho(E_{-i}) = q^{-1}K_iE_i, \quad \rho(K_i^{\pm 1}) = K_i^{\pm 1}.$$

Let $(-, -)_{\text{Lus}}$ also denote the $\mathbb{Z}[q, q^{-1}]$ bilinear inner product on $V_S^{\mathbb{Z}}$ obtained by taking factorwise the above form on $V_{s_i}^{\mathbb{Z}}$, for $i = 1, \dots, \ell(S)$.

Before we can compute the inner product of the elementary tensors, we first have to compute $(-, -)_{\text{Lus}}$ on $V_+^{\mathbb{Z}}$ and $V_-^{\mathbb{Z}}$. Let e_1^+ be the highest weight vector of V_+ , of weight $(1, 0)$, and define

$$e_0^+ := E_{-1}(e_1^+) \quad \text{and} \quad e_{-1}^+ := E_{-2}(e_0^+).$$

Note that e_0^+ and e_{-1}^+ are of weight $(-1, 1)$ and $(0, -1)$ respectively. Similarly, let e_1^- be the highest weight vector of $V_-^{\mathbb{Z}}$, of weight $(0, 1)$, and define

$$e_0^- := E_{-2}(e_1^-) \quad \text{and} \quad e_{-1}^- := E_{-1}(e_0^-).$$

Note that e_0^+ and e_{-1}^+ are of weight $(1, -1)$ and $(-1, 1)$ respectively. Using the rules above, we get

$$(e_i^\pm, e_j^\pm)_{\text{Lus}} = \delta_{ij}.$$

On $V_S^{\mathbb{Z}}$, we now get

$$(5.3.10) \quad (e_{J'}^S, e_{J''}^S)_{\text{Lus}} = \delta_{J', J''},$$

for any elementary tensors $e_{J'}^S$ and $e_{J''}^S$.

Note that both $(-, -)_{\text{Lus}}$ and $(-, -)_{\text{Kup}}$ are $\mathbb{Z}[q, q^{-1}]$ bilinear and symmetric. Therefore, in order to show that they are equal, it suffices to show that we have

$$(w_J^S, w_J^S)_{\text{Lus}} = (w_J^S, w_J^S)_{\text{Kup}},$$

for any $w_J^S \in B_S$.

Let $w_J^S \in B_S$ be arbitrary and write

$$w_J^S = e_J^S + \sum_{J' < J} c(S, J, J') e_{J'}^S,$$

as in Theorem 2.1.5. Then, by (5.3.10), we get

$$(5.3.11) \quad (w_J^S, w_J^S)_{\text{Lus}} = 1 + \sum_{J' < J} c(S, J, J')^2.$$

Finally, let us compute $(w_J^S, w_J^S)_{\text{Kup}}$. By (5.3.9), we see that

$$(w_J^S, w_J^S)_{\text{Kup}} = \langle w_J^S, w_J^S \rangle = q^{\ell(S)} \langle (w_J^S)^* w_J^S \rangle_{\text{Kup}}.$$

The first equality follows from Proposition 5.3.21. Now consider the way in which the coefficients $c(S, J, J')$ change under the symmetry $x \mapsto x^*$, for x any Y , cup or cap with flow. Comparing the corresponding weights in (2.1.11) and (2.1.12), we get

$$\text{weight}(x^*) = q^{-(\ell(t(x)) - \ell(b(x)))} \text{weight}(x).$$

where $t(x)$ and $b(x)$ are the top and bottom boundary of x . Recall also that the canonical flow on w_J^S has weight 0 (see Lemma 2.1.4). It follows that

$$\begin{aligned} (w_J^S, w_J^S)_{\text{Kup}} &= q^{\ell(S)} \langle (w_J^S)^* w_J^S \rangle_{\text{Kup}} \\ &= q^{\ell(S)} \left(q^{-\ell(S)} + q^{-\ell(S)} \sum_{J' < J} c(S, J, J')^2 \right) \\ &= 1 + \sum_{J' < J} c(S, J, J')^2. \end{aligned}$$

This finishes the proof that $(-, -)_{\text{Lus}} = (-, -)_{\text{Kup}}$. □

APPENDIX A. FILTERED AND GRADED ALGEBRAS AND MODULES

In this appendix, we have collected some basic facts about filtered algebras, the associated graded algebras and the idempotents in both. Our main sources are [53] and [54]. In this appendix, everything is defined over an arbitrary commutative associative unital ring K .

Let A be a finite-dimensional associative unital K -algebra with an increasing filtration of K -submodules

$$\{0\} \subset A_{-p} \subset A_{-p+1} \subset \cdots \subset A_0 \subset \cdots \subset A_{m-1} \subset A_m = A.$$

Actually, for any $t \in \mathbb{Z}$ we have a subspace A_t , where we extend the filtration above by

$$A_t = \begin{cases} \{0\} & \text{if } t < -p, \\ A & \text{if } t \geq m. \end{cases}$$

Note that in the language of [53], such a filtration is *discrete, separated, exhaustive and complete*. If $1 \in A_0$ and the multiplication satisfies $A_i A_j \subseteq A_{i+j}$, we say that A is an associative unital *filtered algebra*. The *associated graded algebra* is defined by

$$E(A) = \bigoplus_{i \in \mathbb{Z}} A_i / A_{i-1},$$

and is also associative and unital. Although A and $E(A)$ are isomorphic K -modules, they are not isomorphic as algebras.

A finite-dimensional *filtered A -module* is a finite-dimensional unitary A -module M with an increasing filtration of K -submodules

$$\{0\} \subset M_{-q} \subset M_{-q+1} \subset \cdots \subset M_t = M,$$

such that $A_i M_j \subseteq M_{i+j}$, for all $i, j \in \mathbb{Z}$, after extending the finite filtration to a \mathbb{Z} -filtration as above.

We define the t -fold *suspension* $M\{t\}$ of M , which has the same underlying A -module structure, but a new filtration defined by

$$M\{t\}_r := M_{r+t}.$$

Given a filtered A -module M , the *associated graded module* is defined by

$$E(M) := \bigoplus_{i \in \mathbb{Z}} M_i / M_{i-1}.$$

An A -module map $f: M \rightarrow N$ is said to *preserve the filtrations* if $f(M_i) \subseteq N_i$, for all $i \in \mathbb{Z}$. Any such map $f: M \rightarrow N$ induces a grading preserving $E(A)$ -module map $E(f): E(M) \rightarrow E(N)$ in the obvious way.

This way, we get a functor

$$E: A\text{-mod}_{\text{fl}} \rightarrow E(A)\text{-mod}_{\text{gr}},$$

where $A\text{-mod}_{\text{fl}}$ is the category of finite-dimensional filtered A -modules and filtration preserving A -module maps and $E(A)\text{-mod}_{\text{gr}}$ is the category of finite-dimensional graded $E(A)$ -modules and grading preserving $E(A)$ -module maps.

Recall that $A\text{-mod}_{\text{fl}}$ is not an abelian category, e.g. the identity map $M \rightarrow M\{1\}$ is a filtration preserving bijective A -module map, but does not have an inverse in $A\text{-mod}_{\text{fl}}$. In order to avoid such complications, one can consider a subcategory with fewer morphisms. An A -module map $f: M \rightarrow N$ is called *strict* if

$$f(M_i) = f(M) \cap N_i$$

holds, for all $i \in \mathbb{Z}$. Let $A\text{-mod}_{\text{st}}$ be the subcategory of filtered A -modules and strict A -module homomorphisms.

Lemma A.0.23. *The restriction of E to $A\text{-mod}_{\text{st}}$ is exact.*

We also need to recall a simple result about bases. A basis $\{x_1, \dots, x_n\}$ of a filtered algebra A is called *homogeneous* if, for each $1 \leq j \leq n$, there exists an $i \in \mathbb{Z}$ such that $x_j \in A_i \setminus A_{i-1}$. In that case, $\{\bar{x}_1, \dots, \bar{x}_n\}$ defines a homogeneous basis of $E(A)$, where $\bar{x}_j \in A_i/A_{i-1}$. In order to avoid cluttering our notation, we always write \bar{x}_j and then specify in which subquotient we take the equivalence class by saying that it belongs to A_i/A_{i-1} .

Given a homogeneous basis $\{y_1, \dots, y_n\}$ of the associated graded $E(A)$, we say that a homogeneous basis $\{x_1, \dots, x_n\}$ of A *lifts* $\{y_1, \dots, y_n\}$ if $\bar{x}_j = y_j \in A_i/A_{i-1}$ holds, for each $1 \leq j \leq n$ and the corresponding $i \in \mathbb{Z}$. The result in the following lemma is well-known. However, we could not find a reference in the literature, so we provide a short proof here.

Lemma A.0.24. *Let A be a finite-dimensional filtered algebra and $\{y_1, \dots, y_n\}$ be a homogeneous basis of $E(A)$. Then there is a homogeneous basis $\{x_1, \dots, x_n\}$ of A which lifts $\{y_1, \dots, y_n\}$.*

Proof. We prove the lemma by induction with respect to the filtration degree q . Suppose $A_q = 0$, for all $q < -p$, and $A_q = A$, for all $q \geq m$. Then $E(A_{-p}) = A_{-p}$. Since $\{y_1, \dots, y_n\}$ is a homogeneous basis of $E(A)$, a subset of this basis forms a basis of A_{-p} .

For each $-p+1 \leq q \leq m$, choose elements in A_q which lift the homogeneous subbasis of $E(A_q)$. We claim that the union of the sets of these elements, for all $-p \leq q \leq m$, form a homogeneous basis of A which lifts $\{y_1, \dots, y_n\}$. Call it $\{x_1, \dots, x_n\}$. By definition, the x_j lift the y_j , for all $1 \leq j \leq n$. It remains to show that the x_j are all linearly independent. This is true for $q = -p$, as shown above.

Let $q > -p$ and suppose that the claim holds for $\{x_1, \dots, x_{m_{q-1}}\}$, the subset of $\{x_1, \dots, x_n\}$ which belongs to A_{q-1} . Let

$$\{x_1, \dots, x_{m_q}\} = \{x_1, \dots, x_{m_{q-1}}\} \cup \{x_{m_{q-1}+1}, \dots, x_{m_q}\}$$

be the subset belonging to A_q . Suppose that

$$(A.0.12) \quad \sum_{j=1}^{m_q} \lambda_j x_j = 0,$$

with $\lambda_j \in K$. Then we have

$$\sum_{j=1}^{m_q} \lambda_j \bar{x}_j = \sum_{j=m_{q-1}+1}^{m_q} \lambda_j \bar{x}_j = \sum_{j=m_{q-1}+1}^{m_q} \lambda_j y_j = 0 \in A_q/A_{q-1}.$$

By the linear independence of the y_j , this implies that $\lambda_j = 0$, for all $m_{q-1} + 1 \leq j \leq m_q$. Thus, the linear combination in (A.0.12) becomes

$$\sum_{j=1}^{m_{q-1}} \lambda_j x_j = 0.$$

By induction, this implies that $\lambda_j = 0$, for all $1 \leq j \leq m_{q-1}$.

This shows that $\lambda_j = 0$, for all $1 \leq j \leq n$, so the x_j are linearly independent. □

For a proof of the following proposition, see for example Proposition 1 in the appendix of [54].

Proposition A.0.25. *Let M and N be filtered A -modules and $f: M \rightarrow N$ a filtration preserving A -linear map. If $E(f): E(M) \rightarrow E(N)$ is an isomorphism, then f is an isomorphism (and therefore strict too).*

The most important fact about filtered projective modules and their associated graded projective modules, that we need in this paper, is Theorem 6 in [53].

Theorem A.0.26 (Sjödin). *Let P be a finite-dimensional graded projective $E(A)$ -module. Then there exists a finite-dimensional filtered projective A -module P' , such that $E(P') = P$. Moreover, if M is a finite-dimensional filtered A -module, then any degree preserving $E(A)$ -module map $P \rightarrow E(M)\{t\}$, for some grading shift $t \in \mathbb{Z}$, lifts to a filtration preserving A -module map $P' \rightarrow M\{t\}$.*

Theorem A.0.26 and Proposition A.0.25 immediately imply the following result.

Corollary A.0.27. *Any complete set of pairwise non-isomorphic graded indecomposable projective $E(A)$ -modules can be lifted to a complete set of pairwise non-isomorphic filtered indecomposable projective A -modules.*

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M.M.: *CAMGSD, Instituto Superior Técnico, Lisboa, Portugal; Universidade do Algarve, Faro, Portugal*
email: mmackaay@ualg.pt

W.P.: *Courant Research Center “Higher Order Structures”, University of Göttingen, Göttingen, Germany;*
Saint Mary’s College of California, Moraga, USA
email: wpl@stmary-ca.edu

D.T.: *Courant Research Center “Higher Order Structures”, University of Göttingen, Göttingen, Germany;*
Mathematisches Institut, Georg-August-Universität Göttingen, Göttingen, Germany
email: dtubben@uni-math.gwdg.de